Conflict set - Parabola

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Target knowledge	Realizing that the parabola defined as a locus of points equidistant from a horizontal line and a given point is a graph of some quadratic function.
Broader goals	Calculating using coordinates, connecting geometrical and algebraic representation of a mathematical object
Prerequisite mathematical knowledge	Coordinate system in the plane, equation of the line, the distance formula (or Pythagoras theorem) Building on Conflict lines - introduction
Grade	Age 16-17, Grade 10 (After teaching quadratic functions)
Time	70 minutes
Required material	Grid paper with coordinate system and/or ICT if possible, to plot and construct
Problem: Consider line *p* with the equation $y=2$ and point $A(5,4)$. Show that points $M(7,4)$ and $N(1,7)$ are equally distant from line *p* and point *A*. Find all points with the same property!	

Phase	Teacher's actions incl. instructions	Students' actions and reactions	Observations from implementation
Devolution (didactical) 2 min	Teacher recalls a navigation problem from the previous scenario. He poses the problem: one student is standing in the class and the others form a navigation route for a robot to avoid the student and the closest wall in the class.	Students organize the scenery.	
Action (adidactical) 5 min	Teacher observes the students' reaction and reasoning how they know they are at equal distance to the student and the wall.	Students are trying to form the route by standing at equal distance to the specific student and the wall.	

Devolution (didactical) 3 min	Teacher presents a mathematical model of this specific situation: Consider line *p* (representing the wall) with the equation $y=2$ and point $A(5,4)$ (representing the student). Show that points $M(7,4)$ and $N(1,7)$ are equally distant from line *p* and point *A*. Find all points with the same property!	Students analyse the problem and start organizing the data.	
Action (adidactical) 7 min	Observes if students remember the distance formula or calculate the distance using Pythagoras theorem.	Students make some sketches of the problem. They calculate the distances of points *M* and *N* to point *A* and line *p*. Verify the statement. Think of strategies to find more such points. Students are expected to try to draw or construct some point in the coordinate system. Expected answer: Finding the point $(5,3)$ since it is halfway between point *A* and line *p*. Finding points $(3,4)$ and $(9,7)$ because of the symmetry. Finding all points $T(x,y)$ such that $d(T,A)=d(T,p)$.	

Formulation (adidactical) 5 min	Teacher circulates in the classroom to identify what different ideas the students recall and use.	Students discuss in groups what they did, how many different points they found.	
Validation (didactical and adidactical) 4 min	Teacher asks certain students to present what they did so far.	Students give arguments why their point are relevant by Pythagoras, symmetry.	
Devolution (didactical) 2 min	If none of the groups (or just 1-2) is working further than finding two or three points, teacher must take the initiative. Teacher sets a motivating question: Can you find a point on line $y=12$ the that is equidistant from line *p* and point *A*? And on line $x=6.4$?	Students listen.	
Action (adidactical) 20 min	Teacher circulates in the classroom	Student try to solve the problem by plotting the line $y=12$. They realise that the distance to line *p* is 10, a point on line $y=12$ has coordinates $T(x,12)$ and set the equation $d(T,A)=10$. They find two points and can conclude that is because of the symmetry. Some of the students might realize (without knowing the point-line formula) that the distance of any point $T(x,y)$ to line $y=2$ is $\|y-2\|$ and then try to find all points $T(x,y)$ such that $d(T,A)=\|y-2\|$.	

		The others continue investigating calculating distances for different points on line $x = 6.4$. Or by estimation approx. $(6.4, 3.5)$. Or students recall the concept of perpendicular bisector and construct it geometrically (possible use of ICT). Some students might notice the parabolic shape and work with equation $y = a(x-h)^2 + k$, or $y = ax^2 + bx + c$.	
Formulation (adidactical) 5 min	Teacher circulates in the classroom to identify what different ideas the students recall and use.	Students discuss in groups what they did, what is the set of all points with the requested property and how to write it down. If they recognized that the solution is a parabola, how to find the coefficients in the equation.	
Validation (didactical and adidactical) 10 min	Teacher asks certain students to present what they did so far.	Students give arguments why their equation is the right one. They can prove that all points that satisfy the equation are equidistant to the given line and the given point.	
Institutionalisation (didactical) 7 min	Teacher highlights the similarities and differences in the students' strategies and explains how all the information that students produced can be mathematically expressed. If students assumed that the set of all points with the given property is a parabola and found its equation, they have to prove that it satisfies the equal distance property.	Students listen and recognize their own strategy as one of those mentioned by the teacher.	

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<td>Possible ways for students to realize target knowledge</td>
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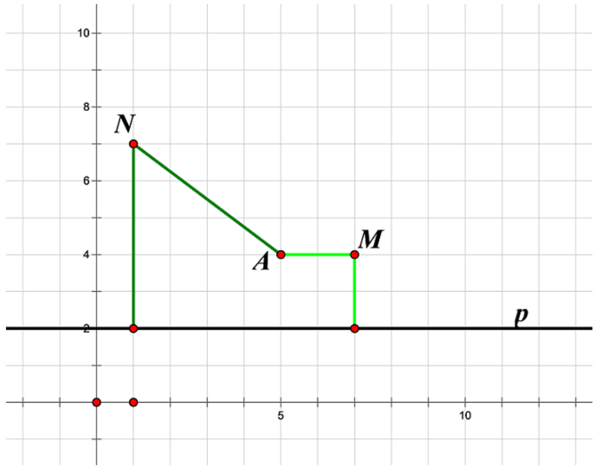
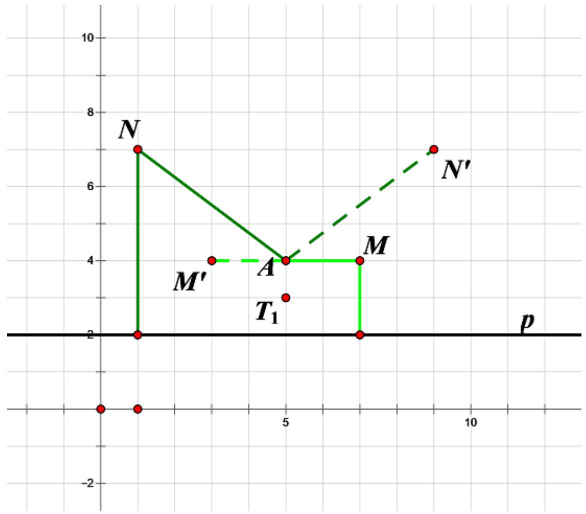
Calculating distance between M, N and A, p
$d(M,p)=2$
$d(M,A)=\sqrt{(5-7)^2+(4-4)^2}=2$
$d(N,p)=5$
$d(N,A)=\sqrt{(5-1)^2+(4-7)^2}=5$

Students might read the distances from the coordinate system.


Students could also find three symmetrical points M’, N’ and T_1.
If students are familiar with the point-line distance formula, they can use it in finding all points $T(x,y)$ such that $d(T,A)=d(T,p)$, where $A(5,4), p...y-2=0$
$(x-5)^2+(y-4)^2=\left(\frac{|y-2|}{\sqrt{0^2+1^2}}\right)^2$
$x^2-10x+25+y^2-8y+16=y^2-4y+4$
$y=\frac{1}{4}x^2-\frac{10}{4}x+\frac{37}{4}$

Students can also notice that the distance is equal to $|y-2|$ just from the sketch, without knowing the point-line formula.


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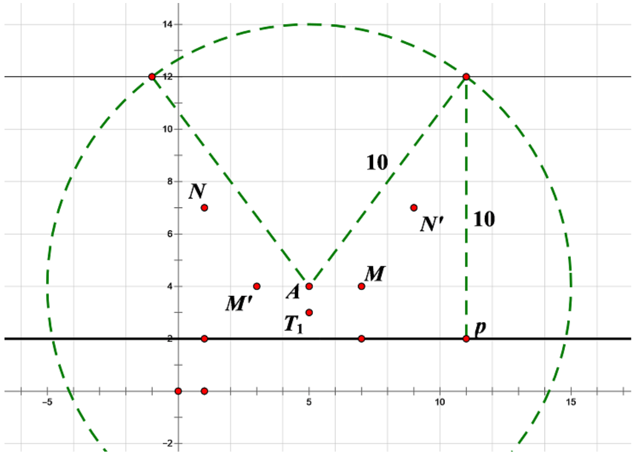
To find a point on line $y=12$ that satisfies the condition, students can conclude (without point-line distance formula) that the distance to line $y=2$ is 10 and calculate

$$(x-5)^2+(12-4)^2=100$$

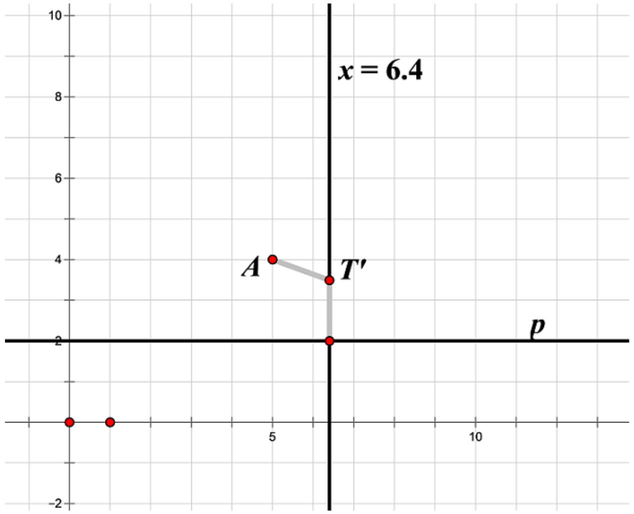
$$x^2-10x-11=0$$

$$x_1=-1, x_2=11$$

They will find two points $(-1,12),(11,12)$

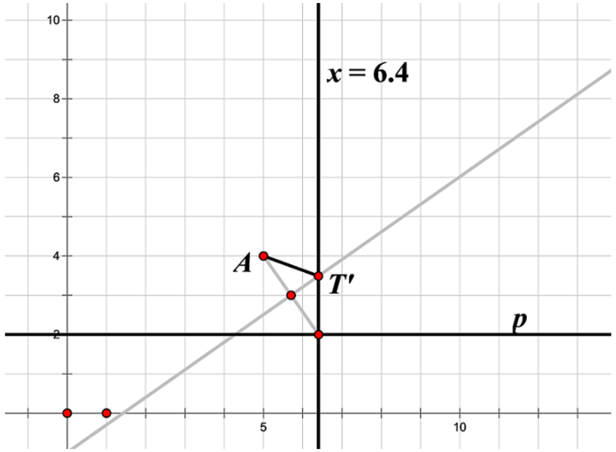


To find a point on line $x=6.4$ students will probably estimate it's coordinates, $T'(6.4,3.5)$



Some students might try to construct the point using perpendicular bisector.

If they assume it is a parabola: $y = ax^2 + bx + c$, they will find a, b, c by substituting points $M(7,4)$, $N(1,7)$ and, for example $M'(3,4)$.



$$a \cdot 7^2 + b \cdot 7 + c = 4$$
$$a \cdot 1^2 + b \cdot 1 + c = 7$$
$$\underline{a \cdot 3^2 + b \cdot 3 + c = 4}$$
$$49a + 7b + c = 4$$
$$a + b + c = 7$$
$$\underline{9a + 3b + c = 4}$$
$$\left.\begin{matrix} 48a + 6b = -3 \\ \underline{40a + 4b = 0} \end{matrix}\right\} \Rightarrow b = -10a \Rightarrow a = \frac{1}{4}, b = -\frac{10}{4}$$
$$\frac{1}{4} - \frac{10}{4} + c = 7 \Rightarrow c = \frac{37}{4}$$
$$y = \frac{1}{4}x^2 - \frac{10}{4}x + \frac{37}{4}$$

Proof that the points satisfying the equation of the parabola have the geometric property; equally distant to the point and the line:

$$(x-5)^2 + \left(\frac{1}{4}x^2 - \frac{10}{4}x + \frac{37}{4} - 4\right)^2 = \left(\frac{1}{4}x^2 - \frac{10}{4}x + \frac{37}{4} - 2\right)^2$$
$$x^2 - 10x + 25 + \frac{1}{16}x^4 + \frac{100}{16}x^2 + \frac{441}{15} - \frac{5}{4}x^3 + \frac{21}{8}x^2 - \frac{105}{4} =$$
$$= \frac{1}{16}x^4 + \frac{100}{16}x^2 + \frac{841}{16} - \frac{5}{4}x^3 + \frac{29}{8}x^2 - \frac{145}{4}x$$
$$0 = 0$$