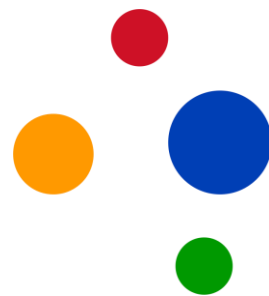




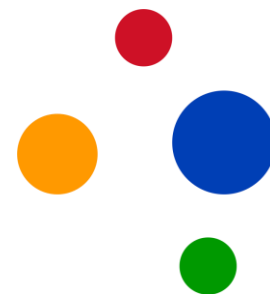
Mathematics Education -
Relevant, Interesting and Applicable

MERIA SCENARIOS AND MODULES





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MERIA SCENARIOS AND MODULES

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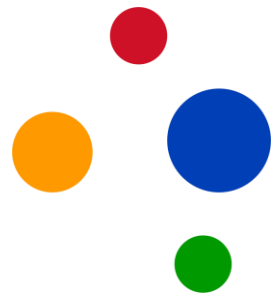
DESIGN AND VISUALS

Irina Rinkovec

Project MERIA, August 2019
www.meria-project.eu

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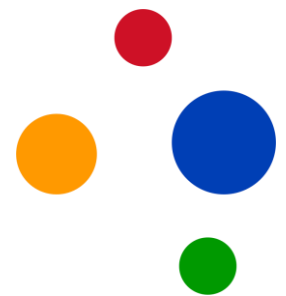
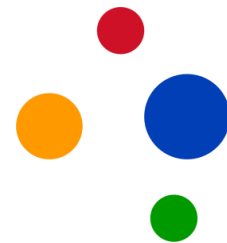


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Introduction

The booklet *MERIA Scenarios and Modules* represents one of the major outcomes of the project MERIA and comprises five teaching scenarios and their corresponding modules. The “model” or pattern for the scenarios and modules has been introduced in *MERIA Template for Scenarios and Modules*, where one more showcase scenario and module were published. The design of these materials is based on the theoretical background presented in the *MERIA Practical Guide to Inquiry-Based Mathematics Teaching*.

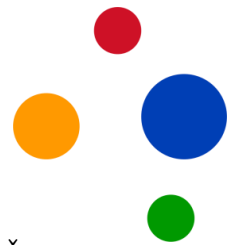
A *scenario* describes a didactical situation, to be realised in a lesson, and the epistemological assumptions and reasoning behind it. It also describes the goals of the situation in terms of curriculum area and specific target knowledge and competency, and it provides a clear structure for the lesson based on the Theory of Didactical Situations. In addition to the scenario, a *module* contains written and digital materials such as student assignments or digital worksheets, developments of the explicit rationale for the choice of the problem(s) and the teaching methods with further perspectives from Realistic Mathematics Education. Furthermore, it also contains elements of the experiences and results which have been collected during the implementations of the scenario, including potential gains and pitfalls for students with specific preknowledge.

The use of a scenario can be a challenging task for a teacher. The basic ideas behind it might not be obvious, and the target knowledge can be difficult to reach. Therefore, the modules make some of the intentions of the authors more explicit and offer support to the teacher by describing variations that could be expected in the implementation of a scenario. Hence, complete modules are published here. The scenarios are published in a more user-friendly version separately on the project’s webpage.

All scenarios contain some potential for students to use ICT for their mathematical inquiry, but the problems can also be addressed without the use of technology. These variations are also described in the scenario or in the module. All additional teaching and learning materials for each scenario are published on the MERIA webpage.

It should be noted that it takes time for the students to get used to inquiry-based learning. The same applies to the teachers, who need to find the balance between an excessive intervention (which ruins the students’ opportunities for inquiry) and leaving the students with too few resources to be able to do meaningful inquiry. It is a strong conviction of the MERIA project team that it would be optimal for a teacher to experience the scenarios in the context of professional development through a series of MERIA workshops face-to-face supported by the MERIA Practical guide to IBMT.

During the period from July 2017 to December 2018, the MERIA project team was developing around ten different scenarios covering different topics that fit into curricula in all partnering countries: Croatia, Denmark, The Netherlands, and Slovenia. In each country, three to four schools were associated with the project to test the scenarios. Many thanks go to our partners in associated schools for their dedicated work. The associated schools are:



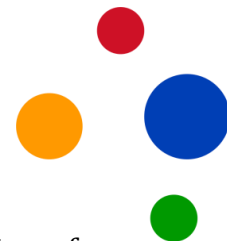
- from Croatia: Gospodarska škola Varaždin, Tehnička škola Požega, Elekstrostrojarska škola Varaždin, XII. gimnazija Zagreb
- from Denmark: ZBC, Next København, Roskilde Katedralskole
- from the Netherlands: Comenius College Hilversium, Hermann Wesseling College, Stedelijk Gymnasium Utrecht
- from Slovenia: Ekonomska šola Novo mesto, Gimnazija Jesenice, Gimnazija Franca Miklošiča Ljutomer

The process of testing the scenarios has led to multiple revisions and provided interesting information that served to improve the scenarios. It was crucial to make the choice of five modules for this booklet as the most relevant (in all countries!) and successful products of the project. The teachers from the associated schools were first presented with the theoretical background through interactive workshops provided by the project members. In this way, the teachers were prepared to work with the scenarios. The classroom implementation of each scenario was observed by project members or another teacher from the same school. The teachers have afterward reflected on the implementation of the lesson based on a questionnaire and reported to the project members orally at the next meeting. Students' work has been documented and students have also filled out a short questionnaire reporting on the extent to which they found the lesson challenging and interesting and stating whether they would like to engage in a similar activity again. Further information about the questionnaires, reports, and methodology is available in the *MERIA project impact analysis*.

The choice of five scenarios, for which complete modules are presented in this booklet, has been made based on the criteria specified at the project meeting in Copenhagen in August 2018: the potential for inquiry and the didactical potential of the scenario, feasibility of the scenario for the students and the teachers, the suitability of the topic in terms of relevancy and applicability, students' reactions, as well as the diversity of topics present in the high school curricula of the partner countries.

Thus, the choice of MERIA scenarios covers the following topics: modelling of a simple business problem by using a piecewise linear function, modelling of how breaking distance depends on speed by using a quadratic function, solving an elementary geometry problem using perpendicular bisectors, reasoning about salary distribution in three companies using mean, mode and median, and modelling a curved object (a slide or a ski jump) as a smooth curve. Our intention was not to cover as many topics as possible in the (shared) high school curriculum but to produce showcase examples that support teachers in establishing inquiry-based learning in their classrooms. We find these scenarios suitable for the development of mathematical modelling, progressive formalization, conjecturing and proving, scientific approach, encouraging understanding instead of memorizing, critical thinking, autonomous inquiry and applications to real-world problems.

We end this introduction by describing each of the five scenarios in terms of the main problem posed to the students and the target knowledge that is aimed at by the scenario.



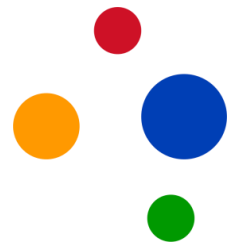
In the first scenario, the students are asked to consider data about the production of bicycles and the construction of a factory in four different locations, in order to advise a company on how to find the optimal location, depending on the expected production. The production in each location may be modelled by a linear function and the students can develop different strategies to compare the locations. Students use graphical representations and very often some kind of software, think critically and summarize their findings to write a report for the company.

The second scenario asks the students to study the dependence of the braking distance of a car on the initial speed at the beginning of braking. The dependence is quadratic, which is a new type of function for students (expected students' prerequisite is assumed to be knowledge of linear functions). Thus, the scenario serves as an introduction to quadratic functions, improves numerical skills and reasoning, but also draws conclusions and reflects on everyday situations with a sense of responsibility.

In the third scenario, students are given a map of a desert with six wells and are asked to divide the map into regions depending on the proximity of points to the wells. To solve the problem (construct the so-called Voronoi diagrams) it is necessary to use perpendicular bisectors. Students may use specially designed applets to explore the problem and to construct further similar situations that lead to the investigation of cyclic configurations of points.

The fourth scenario focuses on simple statistical reasoning about a data set. The data represents the salaries of employees in different companies and the students' task is to analyse the data and reach a conclusion about where they would prefer to be employed. Students are expected to reach the descriptors of central tendencies, such as mean and median, however, their analysis may easily lead to other points of view, such as graphical representations of percentiles, etc.

The fifth scenario is about the construction of a slide that consists of a curved and straight part, which should be connected in a smooth way - the point of the scenario is that "smooth" has a precise mathematical definition. The students are merely asked to construct the slide in such a way that provides a comfortable ride. Hence, the task is to analyse in what way to connect the two parts and to discover that the straight part should be tangent to the curved part at the point of contact. Students may choose various curves for the curved path to start with and then take many different strategies to construct the tangent. In the case of quadratic curves, the problem may be solved in an elementary way, but for other curves, the problem leads the students to the idea of the derivative of a function.



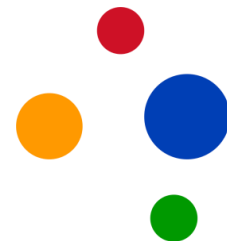
MERIA Module “Bicycle factory”

Linear and piecewise linear functions

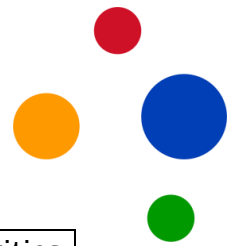
The teaching scenario

Target knowledge	The construction of piecewise linear functions as an optimal solution to a problem with multiple linear conditions.																
Broader goals	<p>Drawing graphs of (linear) function on paper and using ICT. Discussion about scaling the graphs along one axis. A deeper understanding of linear functions (the slope a and the constant b) by using them on linear conditions to construct piecewise linear functions. Discussion of the continuous and discrete aspects in relation to algebraic and graphical representations in the modelling process.</p> <p>Inquiry skills: experimenting with numbers before drawing graphs, disregarding unimportant data and obvious suboptimal factories, interpreting the results obtained in the modelling process, taking responsibility for the final report and presenting findings in a form of advice.</p> <p>Interdisciplinary skills: students may discuss various economic aspects of the problem such as the difference between profit/earnings and revenue. Professional communication skills are emphasized in writing the report.</p>																
Prerequisite mathematical knowledge	Drawing the graph of a linear function. Familiarity with the notation $f(x) = ax + b$ and the interpretation of a and b .																
Grade	Age 15 - 16, grade 9 - 10 (even earlier with smaller numbers)																
Time	50 min (80 min)																
Required material	<p>The table with data about costs</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th>Areas of location</th> <th>Costs of building the factory in that area in €</th> <th>The costs of producing one bicycle at the factory in €</th> </tr> </thead> <tbody> <tr> <td>A</td> <td>300 000</td> <td>120</td> </tr> <tr> <td>B</td> <td>450 000</td> <td>110</td> </tr> <tr> <td>C</td> <td>660 000</td> <td>60</td> </tr> <tr> <td>D</td> <td>680 000</td> <td>80</td> </tr> </tbody> </table> <p>Grid paper and/or applets (for changing the linear conditions) and/or ICT in general, for plotting, changing and adding conditions, finding intersections, etc. A wide black or whiteboard (or smartboard).</p>		Areas of location	Costs of building the factory in that area in €	The costs of producing one bicycle at the factory in €	A	300 000	120	B	450 000	110	C	660 000	60	D	680 000	80
Areas of location	Costs of building the factory in that area in €	The costs of producing one bicycle at the factory in €															
A	300 000	120															
B	450 000	110															
C	660 000	60															
D	680 000	80															
Problem:	<p>You are a consultant who advises companies on where to run factory buildings for the production of bicycles. Based on the table showing the costs in different areas, what would you advise the companies and why?¹</p>																

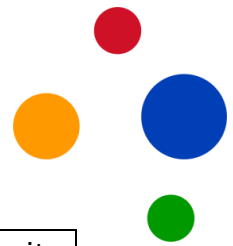
¹ The problem is inspired by Example 2.10 discussed in the book „Primijenjena matematika podržana računalom“, designed also by the author of this scenario in the scope of the project „STEM genijalci“.



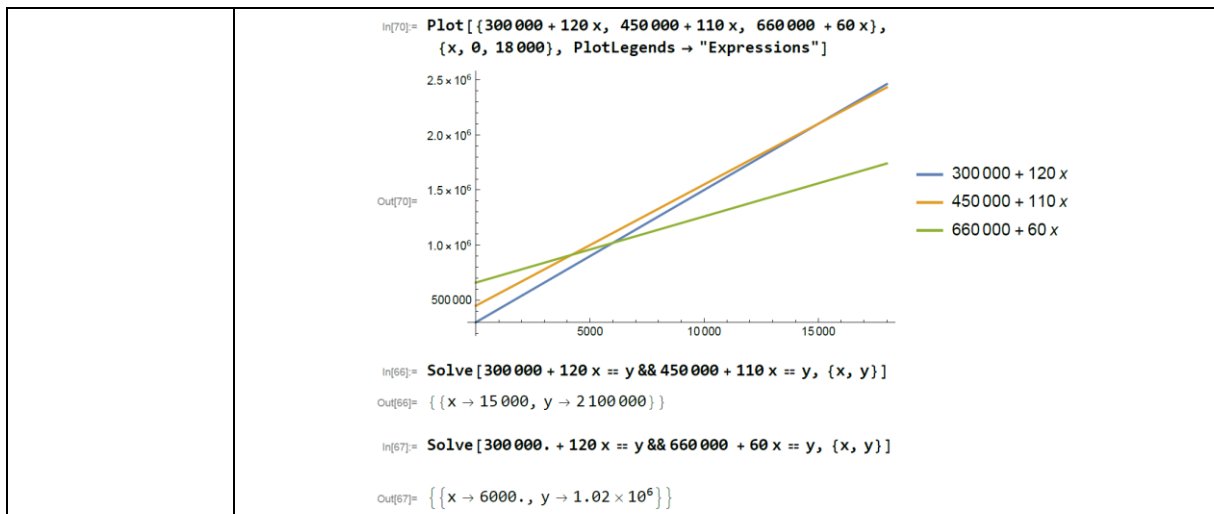
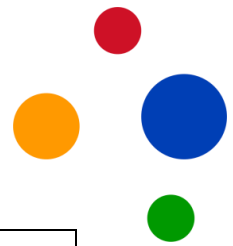
Phase	Teacher's actions incl. instructions	Students' actions and reactions
Devolution (didactical) 5 minutes	The teacher explains the situation and the table above and poses the problem. "How would you in general guide the companies to place their factory? You should work with your neighbour and prepare to present your solution on the board later on."	Students listen, understand the relevance of the problem and feel engaged to work on it. They may have questions as to the meaning of the table and the problem. The teacher should explicitly give the students a chance to ask such questions, to make sure everyone understands the task.
Action (adidactical) 15 (20) minutes	The teacher observes and notes how students approach the problem. Here the teacher gains knowledge about the students' prerequisite knowledge. It is important that the teacher does not give "hints" to the pairs, and avoids interaction with them except, if needed, to repeat the assignment.	The pairs start trying out different strategies or ideas based on their prerequisite knowledge. See "Possible ways for students to realize target knowledge" below. Because the students work in pairs, the adidactic formulation will occur.
Formulation (didactical) 10 (15) minutes	The teacher chooses groups (at least 5) to present different strategies at the black/whiteboard - the board should be divided into areas before the presentations. The students are not allowed to erase afterward. Then, let the chosen pairs present orally, starting with simpler solutions. At this point, no validation is sought for.	Pairs are presenting accordingly to the teacher's plan (first, simple solutions based on numbers, then solutions with graphs and functions).
Devolution (didactical) 1 minute	Discuss with your partner what similarities or differences you see in the presented work. Use this to improve your own answer to the management of the factory. I will ask you to report back after 5 (10) minutes.	Students listen. The teacher should make sure the students understand.
Action / formulation (adidactical) 5 (15) Minutes	The teacher circulates in the classroom to observe what the pairs had noticed and discussed, and how they make use of others' ideas.	The pairs are pointing to similarities and differences, trying to improve their own solution.



<p>Formulation and validation (didactical)</p> <p>10 (15) minutes</p>	<p>The teacher calls on different pairs, to get as many observations and improved answers as possible. The teacher strives to have students identify any mistakes in previous solutions.</p>	<p>Students formulate similarities and differences and explain how they have improved their own solution by taking into account the work of the others; they may also point to shortcomings in some of this work.</p>
<p>Institutionalisation (didactical)</p> <p>5 (10) minutes</p>	<p>The teacher emphasises that there is not one correct answer, but the solution depends on how many bicycles are produced. The teacher first bases explanations on students' solutions on the board, then he/she introduces the notation of functions defined piecewise, using the example :</p> $f(x) = \begin{cases} 120x + 3 \cdot 10^5, & x \leq a \\ 60x + 6.6 \cdot 10^5, & x \geq a \end{cases}$ <p>where $a=6000$.</p> <p>(S)He uses this to summarize how to advise the company: areas B and D are never optimal, while A and C are optimal for the production below and above 6000 bicycles, respectively. The optimal cost function is a piecewise linear function (defined on positive integers).</p>	<p>Students listen and recognize their own strategy in relation to the definition, and reflect on how this compares with the others. They write their notes.</p>
<p>Possible ways for students to realize target knowledge</p>	<ul style="list-style-type: none"> • Some students begin working with some of the numbers, to see what they mean, such as: <ul style="list-style-type: none"> ○ Some students begin by calculating the price for concrete numbers of bicycles in each area. They may use trial and error to find numbers for which two areas give the same. ○ Students can create tables for each geographical area calculating the total costs for each number of bicycles comparing and point to the cheapest solution for any given number of bicycles (this can be done with pen and paper or in a spreadsheet environment). ○ Considering two locations, to cover the difference between the fixed costs with the difference between variable costs (e.g., to answer: how many bicycles must be produced before B is better than A); a total of six such comparisons are needed to provide a complete answer. 	



- Some students take the function approach right away, and write down four equations, where each function represent the yearly costs of the production of x bicycles:
$$f(x) = 120x + 300\,000,$$
$$g(x) = 110x + 450\,000,$$
$$h(x) = 60x + 660\,000,$$
$$k(x) = 80x + 680\,000.$$
 - The graphs of the functions are drawn in one or more coordinate systems, and from the graphic representation, students argue for the placement of the factory.
 - Students who use grid paper might read the point of intersection on coordinate axes.
 - Students who use ICT may plot all the linear functions right away but will struggle to adjust the axis to “see” them.
 - In any case, the interpretation of the functions and the need to minimize the costs do not fall out automatically from any of the above but require thinking about the problem. Mistakes will occur, such as confusing production costs with the selling price or profit, etc.
 - Based on the functional equations, the intersections between functions are found by equating pairs of equations. Students will make use of the graphic representation to know which pairs of equations are relevant. This strategy also requires equation-solving techniques.
- The students may reach different conclusions.
 - Whether the students work with numbers (and tables) or functions (and graphs) some will realize that there is not one “best area”, but the advice to be given depends on how many bicycles are produced. The conclusion could be more or less precise, formulated in words, equations, graphs, etc.
 - Some students will provide a quick and erroneous answer, e.g. “A is best because when we compute the cost to make 1, 2, ..., 10 bicycles, we always get the cheapest price there.”
- **Example of graphs and equations which students could produce** (either on paper or, as here, using technology), to be used to identify how different areas are more economical at different values for the number of bicycles produced.



Explanation of materials

The story about consulting and the table with costs is meant to engage the students in the devolution phase. The table can be given as a handout, or on the black or whiteboard (or smartboard), also the table could be given in a powerpoint presentation, downloaded to computer or smartphone, and so on.

Students in some countries are familiar with the idea of modelling, but in some other countries, they are not. If needed one can take more time to clarify the data from the table. Students can use mobile phones, graphical calculators, GeoGebra, Wolfram Alpha, grid paper, ruler and/or ICT in general, for plotting, changing and adding conditions, finding intersections, etc. A wide black or whiteboard (or smartboard) or posters are needed to enable all student presentations to be shown at the same time (and remain visible to the end of the lesson), as well as additional space for the teachers' final institutionalisation.

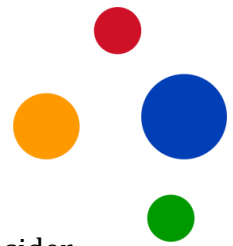
Variations based on didactic variables

The focus in the didactical phases should be on students' formulations (first) and then validation of them. Solutions should not be hinted in didactic phases. In this section, we discuss what could be changed in the above (didactic variables).

The teacher should explain to students that this financial model is simplified and we neglect many factors. In fact, models are usually reductive. In our computation we consider:

- a) costs of building the factory in that area,
- b) costs of producing one bicycle at the factory.

According to standard definitions, there are other fixed running costs, even in a situation without any production. The fixed running costs include costs for heating and salaries of employees with permanent positions, etc.; these are all ignored. Costs b) are included in the variable costs depending on the production quantity, including raw materials, costs of parts of machines which should be replaced, electricity for machine operation, temporary employees' salaries, etc. The problem could be generalized by adding some other costs, but this problem now has only two types of costs.



The consultant report should be based only on the costs a) and b). One may consider saying explicitly that the report should be based only on the given information, while students' own assumptions or estimates could provide a richer set of solutions (cf. the two Danish ones, which are of course somewhat erroneous). Preventing "wrong answers" is not a primary concern, as students can learn from them.

The challenge of the storyline is that the director of the company will decide about the location based on the analysis of the consultant. It is not necessary for the consultant (student) to know if the company plan to produce many bicycles or not, but students sometimes spontaneously make such assumptions.

Upon the decision of the director, the factory will be built just at one location, and it will remain there. There is no option to move the factory.

The milieu: the costs (amount and type) could be chosen differently, but it is perhaps an advantage for beginners not to have many intersections among graphs of minimal costs. Here, there is only one such intersection at $x=6000$. If there are several intersections, which are quite close to each other, the problem becomes more artificial. For example, it is not good if you have two intersection $x_1=5000$ and $x_2=5050$. It would be interpreted that you choose another location because of 50 bicycles.

The products and other elements of the problem could be changed also. Factories producing more than one product, with boundary conditions, will lead to several variables, such as in linear programming.

During the validation, it is important that any wrong strategies or formulas are corrected – as far as possible - by other students. The teacher can engage the rest of the class with questions to certain students like: Can you repeat what was just said? Is it correct? Why do you think so? Where do you know this from? What questions are asked depends on the prerequisites and achievements of the class.

The *length* of the phases should certainly be adapted to students' work.

During the first action phase, students should not be told what to compute or be reminded of specific mathematics to use, such as linear functions.

If the teacher is in doubt whether the students have the prerequisites specified above, the teacher should pose questions such as: How can we compare the costs? Can we neglect any location? Why? etc. The proposed questions should only be posed to groups or individuals if most other students seem to have the required knowledge. The teacher should not lecture each group separately. Further, it is *not needed* to stay with the group until they reach an answer to such a question. Regard it as a minor devolution of a limited problem and let the students act, formulate and validate. Do not support with further questions or hint the answer. If the majority of the class needs to consider those questions, the phase should be shortened and they should be given in plenary; such a need is usually a sign that the initial problem was too difficult or not posed clearly, which one should strive to avoid.



Interventions during the *second phases of action*, formulation, and validation:

The main ideas are similar to those above. If some groups find it very difficult to get started, the teacher can suggest that they compare their strategy with a certain one from another group. This comparable strategy should be well chosen mathematically, meaning that there are some clear relations. This is similar to devolve a new slightly less open sub-problem to the group. If the identification of similarities and differences may be too vague for students, one can also choose to devolve a slightly more specific task, such as: “identify one of the other group’s solutions which you can use to improve your own solution, and do so; then identify an error or shortcoming in one of the solutions, and explain why you disagree.”

During *the final institutionalization*: it is important that most (if not all) strategies shared in the class are addressed and linked to one another. Considering all possible strategies helps the teacher navigate and foresee the students’ inquiry processes. While teaching, do remember that you interact with a dynamic system – students should be allowed to adapt to the milieu, so you cannot expect them all to deliver the same answer!

The inquiry process involves *all* phases. It may take more than one session for the students to fully engage in this kind of teaching. It may be important to stress that we can learn also from alternative or even erroneous solutions.

Some teachers create a schema of possible student strategies, for use during didactic phases. The expected strategies can be listed on a piece of paper and for each strategy; the teacher formulates e.g. three questions that might be fruitful for the teacher to pose when a group presents a given strategy. During the didactical phases, the teacher can note what groups discuss the different strategies and use this to organize the subsequent didactic formulation and validation.

Observations from practice

An important observation from the scenario is that the teachers were trying not to teach the students during all phases of the scenario. This is a nice improvement to keep the didactical potential of the situation. Students had some questions to clarify the problem. Some of them were thinking about the profit, instead of the costs. Some students were confused at the beginning (first devolution) and asked questions about the quality of bicycles, selling price, taxes, a number of produced bicycles ... Some of them realized very fast: *The one with the smallest slope is the cheapest.*

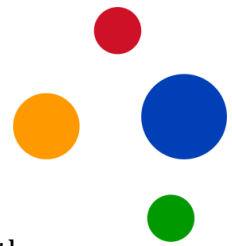
During the action phase, students formulated the following approaches:

I. modelling with linear functions and drawing graphs

- I.1. drawing by hand and calculating intersections as a solution to linear equations;
- I.2. using technology to draw graphs and find intersections (not always correctly).

II. Comparing pairs of areas and analysing the results

- II.1. using linear equations;
- II.2. directly from the table, comparing the fixed costs;



- II.3. other reasoning based on calculations on areas and comparing them, sometimes with self-invented assumptions and errors.

Comparison:

- Approaches I.1 and I.2. were identified similar to the obvious difference in using technology. Students would remark that I.2. is more precise and hence the best solution, while we would offer that I.1. is also a valuable situation because we can detect if our students struggle with drawing the graphs.
- Approach II. requires more logical thinking to reach the conclusion, although the strategy is valid. We have discussed the variants in which students only compare A and C based on their intuition and the necessity to compare with area B.

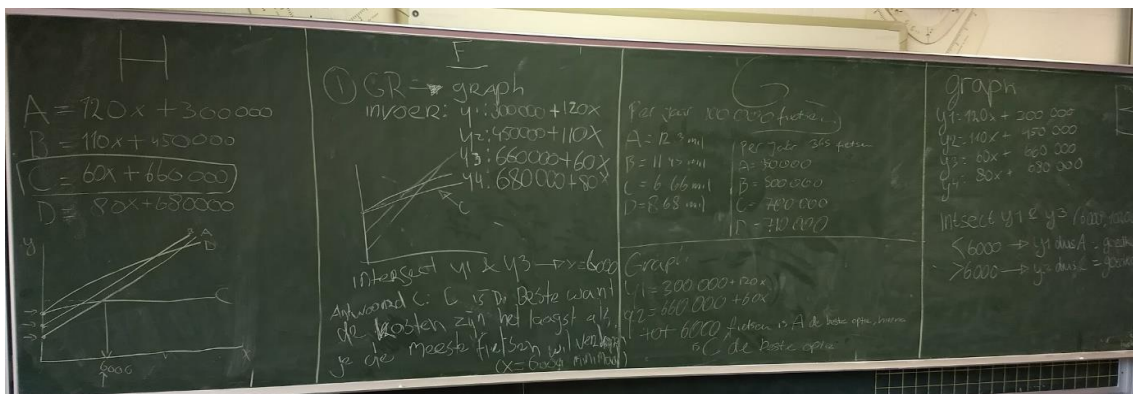
We have pointed out that approach II.2. shows that the problem could be solved without the knowledge of linear functions and their graphs, so the problem might be used to introduce linear functions.

Some groups calculated and compared prices for a particular number of bicycles in each area A, B, C and D. In that case, they had difficulties with formulation because they could not find the exact number of bicycles when the one option starts to be better than the other; sometimes they just made assumptions regarding this. They took approximations or just said that for a smaller number of bicycles is better A, and for the bigger number - option C. One of those groups realized after the second devolution, in action phase, that they could find that number by solving a system of equations.

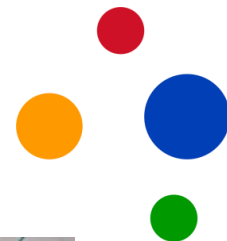
More advanced groups were solving equations and then compared values of the functions on the intervals they had got. Some of the groups used a graphic approach and found intersection points from the graph, using GeoGebra or other software. In this case, calibrating the axis becomes important because of the (large) numbers involved.

Students needed more time for the first devolution and action, but less for the second, so we changed the time according to this remark. Some teachers found they needed to provide some hints or extra questions in the first action and formulation phase. Students did not understand how to compare options without help.

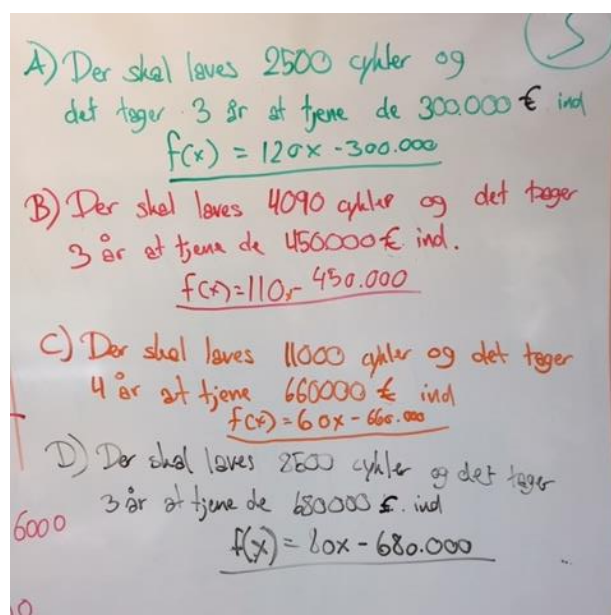
According to the MERIA questionnaire, after this lesson 73.3% of Croatian students agree that mathematics is related to real life, 87% say that the lesson was interesting or much more interesting than usual lecture, and 91.9% of students would like to have such lessons each month.



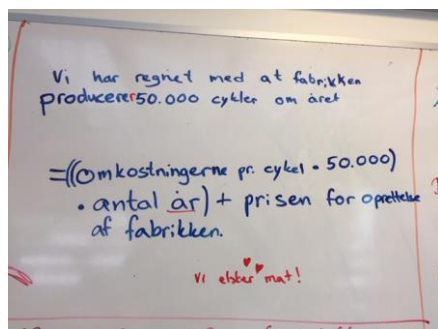
Student group's solutions on the blackboard (Netherlands)



Example of a written presentation from Denmark. The students have considered the four areas separately. They have mistakenly understood “the price for production per bicycle” as “the profit per bicycle”. Then for each area, $f(x)$ is the profit made when producing x bicycles. They have computed, for each area, how many bicycles must be made to cover the cost of building the factory, by finding the root of f .



Literally, for each area, they write, “One should make ... bicycles and it takes ... years to recover ... €”, and the function mentioned. The numbers of years come from the somewhat arbitrary assumptions like “they make at least 2.5 bicycles a day” (oral explanation for case A, none was given for the other areas). It is unclear how they arrived at the number of years in each case except the students explained, for case A, that one could make 2.5 bicycles a day.

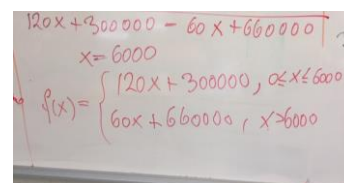


Another group’s solution from the same Danish class: They assume the factory produce 50000 bicycles a year. The formula reads:

$((\text{costs per bicycle} \cdot 50000) \cdot \text{number of years}) + \text{price to build the factory}$

(and at the end, “we love math”). They did not make further conclusions than this formula during the formulation, but applying it would certainly lead to an answer (use area C).

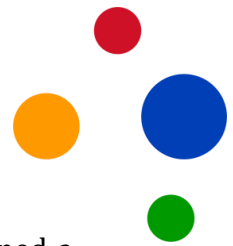
From a teachers’ institutionalization, concerning the solution of the critical equation for the problem, and of piecewise linear functions and the associated notation. In this class, only half of the groups initially found the “good” functions used here.



Evaluation tools

At the end of the lesson or soon after, the following tasks can be used for a quick test of the knowledge students developed during the scenario:

1. Your friend says, “A graph with the smallest slope and the smallest constant corresponds to the cheapest area.” What do you think?
Answer: True, but one does not always have that, as in this case.
2. Your friend says, “The one with the biggest slope and the biggest constant is the most expensive.” What do you think?
Answer: True, but again, we do not have such an area here.



3. You have a very simple situation with data given in the table. You have signed a contract for the production of 5000 bicycles. Which location do you choose?

Areas of location	Costs of building the factory in that area in €	The costs of producing one bicycle at the factory in €
G	0	200
H	300 000	100

Answer: H is cheaper for production bigger than 3000 bicycles.

4. Possible homework: write a document explaining to the director of the company how you would advise him to place his factory.

Suggestion for further problems regarding linear modelling

Include other contexts (e.g. taxi, velocity ...) to apply the target knowledge in other situations (for further institutionalization of the target methods and ideas).

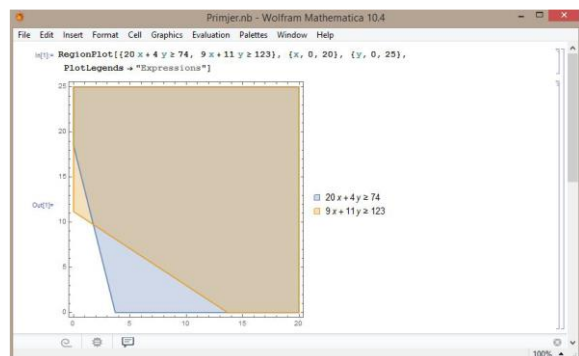
1. Taxi AA start price is 15 € and each kilometre costs 5 €. Taxi BB start price is 20 € and each kilometre costs 4 €. You plan to go 8 kilometres by taxi. Which company would you choose?



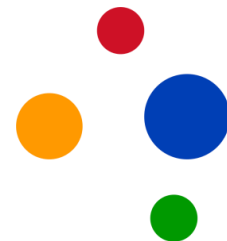
2. Gas price is 0.5 € per m³ for the first 10 m³, then the price decreases for higher gas consumption. Next 20 m³ cost is 0.4 € per m³, then the price drops to 0.3 € per m³. Find the cost function.

3. An athlete should take at least 74 mg of vitamin B and at least 123 mg of vitamin C each day. Multivitamin MM contains 20 mg of vitamin B and 9 mg of vitamin C in 1 g. Multivitamin NN contains 4 mg of vitamin B and 11 mg of vitamin C in 1 g.

What are the minimal doses of multivitamins MM and NN for the athlete to obtain his daily needs? It is not dangerous if he takes a higher dose than needed.



4. Ivana wants to rent a birthday party room for her 17 guests. The price of room RR is 100 € for the room and 10 € additionally for each guest. The price of room PP is 80 € for the room and 12 € additionally for each guest. Which room should Ivana choose?

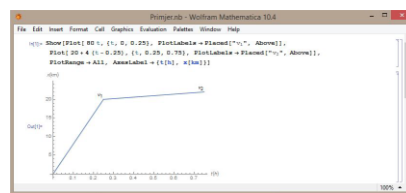


5. The price of one pair of sneakers is 70 €. A company producing the sneakers had an investment of 10000 € to start the production. The production of one pair costs 15 €. Find the profit of the company if they produced 1000 pairs of sneakers.

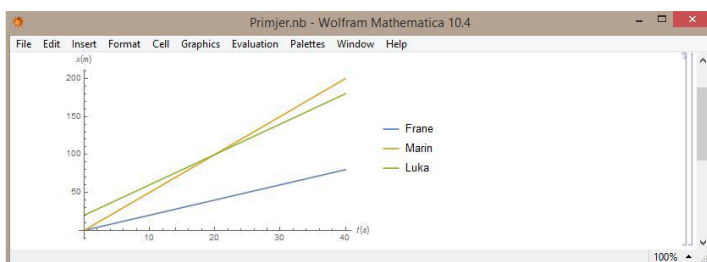


6. A bank offers different rates of interest, depending on the deposit. If you deposit less than 5000 € you obtain 2% rate of interest per year, between 5000 € and 20000 € you obtain 2.2%, and above 20000 € you obtain 2.5%. Find total savings after one year, as a function of the deposit.

7. Anna is driving a car to Zagreb with a constant speed of 80 km/h. After 20 km she ran out of gas, and she walked to the nearest gas station 2 km further down the road. She needed 30 minutes to reach the station. Find the displacement graph with respect to time. Find Anna's average speed. (Graph is a part of the solution)

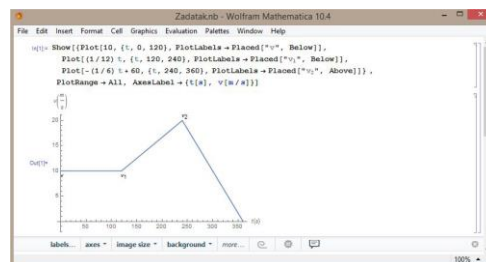


8. Marin and Franck went on a bicycle trip. Luka did not want to wait for them, so he started earlier. See the displacement graph with respect to time.



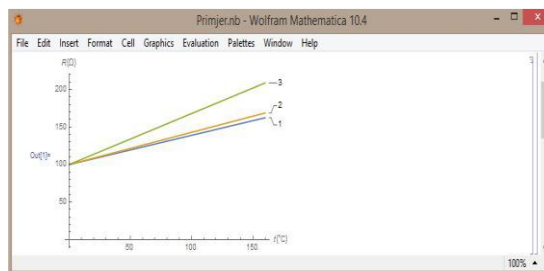
Who is the fastest bicyclist? Who is the slowest? Where does Marin meet Luka? (Graph should be given to students)

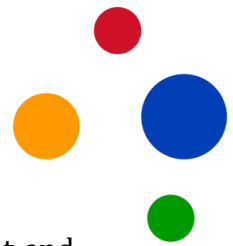
9. Peter rides a motorcycle. First 2 minutes he had constant speed 10 m/s, then after 2 minutes, he achieved the speed 20 m/s with constant acceleration. After that, he started braking and he stopped after 2 minutes. Draw the velocity graph with respect to time.



10. Wire resistance changes with respect to the temperature: $R(T) = R_0(1 + \alpha T)$ where R_0 is the resistance at 0 °C, α the temperature coefficient of resistance and T is the temperature in °C. The resistance for three specific materials at 0 °C is 100 Ω.

Find the temperature coefficients of resistance if the resistances at 100 °C are: 139 Ω (material 1), 143 Ω (material 2) and 168 Ω (material 3).





Find the table with the temperature coefficient of resistance on the internet and say which materials are 1, 2 and 3!

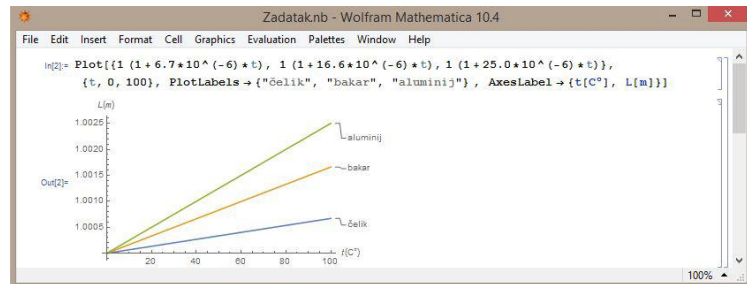
11. Rods are made of different metals, but all of them have length 1 m at 0 °C. Length changes with respect to the temperature: $L(T) = L_0(1 + \alpha T)$, where L_0 is the length at 0 °C, α the coefficient of linear thermal expansion, and T is the temperature in °C. Coefficients of linear thermal expansion are:

Steel $6.7 \cdot 10^{-6} / ^\circ\text{C}$,

Copper $16.6 \cdot 10^{-6} / ^\circ\text{C}$,

Aluminium $25.0 \cdot 10^{-6} / ^\circ\text{C}$.

Find the function of length with respect to temperature for steel, copper, and aluminium.



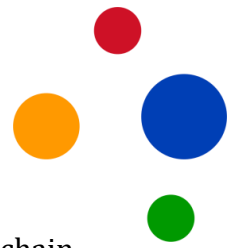
Rationale and RME perspectives on the scenario

The role of contexts in providing opportunities for students to develop (tentative) mathematical ideas is one of the tenets in RME. In this scenario, the factory context is supposed to invite students to create formulas and graphs and to try reasoning with pieces of graphs. This reasoning anticipates the introduction of piecewise defined functions. One can include other contexts (e.g. taxi, velocity, birthday party, renting a space ...) to apply the target knowledge in other situations (for further institutionalization of the target methods and ideas). Learning mathematics in applications is expected to have students develop flexible and applicable mathematical skills.

Relevance and applicability

We consider the following perspectives:

- *Real life and economy*: This knowledge is related to:
 - linear phenomena (taxi costs, phone and internet costs, velocity, birthday party, renting a space, ...)
 - financial modelling (financial models could be linear and nonlinear e.g. company income, profit, average costs, inflation, ...)
 - introduction to optimization
- *Further study*: The knowledge and skills related to this topic are relevant for many disciplines:
 - Linear phenomena show up in science everywhere. Furthermore, for nonlinear problems linearization is a common method for solving the problems, if it is possible. We often compute linear regression and correlation coefficients to produce a linear model and to test the linearity of a data set, even when it is not known to come from a linear relationship.
 - Everybody should be able to organize a budget in daily life in tables with money that you earn and spend on decision-making and future planning. Moreover, doing business is impossible without financial modelling.
 - Company management routinely uses the optimization of processes. Linear programming, also called linear optimization is a method to achieve the best outcome (such as maximum profit or lowest cost in planning, production, transportation) in a mathematical model whose requirements



are represented by linear relationships. Company supply chain management is doing optimization in the transportation of a product or service from supplier to customer.

- One can make an algorithm and a computer program for solving the problem from a scenario or a more general problem.

Inquiry skills

In the Linear scenario students experience the importance of a number of inquiry skills involved in mathematical modelling, transforming data from real life to mathematical language, organizing data, representing the data, finding an optimum, formulating a proposal, collaborating and communicating. The extent to which these skills are explicitly addressed is largely dependent on the way the teacher involves students in feedback on the methods during the validation phases when he or she calls the groups to present. Furthermore, they can be part of the following formulation phase. In those cases, we suggest teachers note the way to make these inquiry skills explicit and provide feedback such that they can be returned to during follow up lessons.

Potential for a sequence of lessons

The Linear scenario could be part of a longer series of lessons on linear phenomena, financial modelling, and linear optimization.

- *Pre-knowledge*: For such a sequence, we expect students to be familiar with linear functions, and linear equations.
- *An introduction*: a context with a rich open problem, such as the one suggested here. Variations of the supplementary problems suggested above could be used in subsequent lessons.

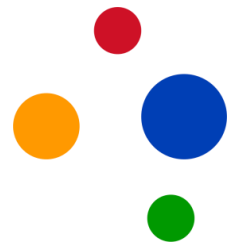
Rationale for the scenario

- *Horizontal mathematizing*: a table showing costs is introduced to discuss the situation. The learners form a first informal model of the situation like $((\text{costs per bicycle} \cdot 50000) \cdot \text{number of years}) + \text{price to build the factory}$, and start using language that anticipates on mathematical optimization methods “costs per bicycle”, “time needed to recover an investment”. This mathematization of the factory context into the world of mathematics provides many opportunities to further develop and institutionalize the target knowledge building on students’ contributions. Students try to discover a solution in groups and prepare a presentation of their findings. The teacher guides the discussion on similarities and differences between these findings to reach the conclusion.
- *Vertical mathematizing*: the mathematics involved in the problem is further developed. Make a general hypothesis or algorithm for finding optimal costs for the given table of data. Furthermore, the model is made more abstract or general, see above for further study.

Conclusion, reflection, and suggestions for further study

The learner reflects, integrates ideas, and makes concepts and skills explicit; the teacher highlights main learning points.

A follow-up lesson could further investigate what the conclusion(s) of the scenario tells us about the initial findings in the groups: What were helpful ideas? Which could be improved? How can we make a general hypothesis or algorithm for finding optimal costs for the given table of data? What were the strategies or ways of working that helped you to come to your results?

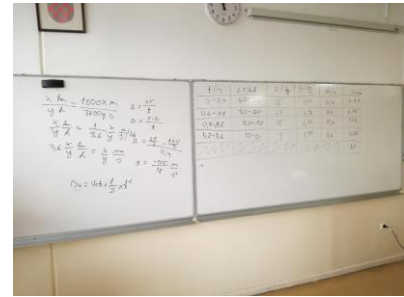


MERIA Module “Braking distance”

Quadratic relationship

The teaching scenario

Target knowledge	Braking distance is quadratically dependent on the initial speed.
Broader goals	<p>Quadratic functions and their characterization by constant second derivative (second differences for quadratic sequences), or by constant decreasing or increasing first derivative (differences for quadratic sequences). Making calculations with different measuring units. Organizing data. Formulating functional relationship (writing the formula for function rule). Drawing graphs of (quadratic) function on paper or using ICT.</p> <p>Inquiry skills: analysing data and looking for patterns in the tables, justifying findings (argumentation) during the presentations (the calculations dominate the process and students have to summarize their approach to others).</p> <p>Interdisciplinary skills: students have to work with variables from physics and make sense of the situation (bridging the two worlds of notations and procedures). Professional communication skills are emphasized in writing the report. The students also discuss the responsibility of drivers and safety in traffic.</p>
Prerequisite mathematical knowledge	Basic knowledge on functions, the relationship between constant speed and distance, average speed, conversion of km/h into m/s (and vice versa)
Grade	Age 16, grade 10 (whenever quadratic functions are introduced)
Time	90 minutes, two lessons
Required material	Handouts with tables to be filled, calculator, computer, graph paper



Presentation from this situation, in Croatia.

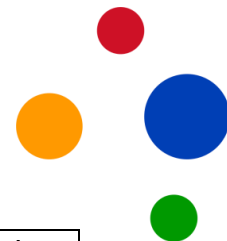
Problem: In a city area with primary schools, parents complain about the set speed limit, considering it inadequate for the area with schoolchildren. A group of reckless drivers says that they do not need to worry because they brake in time. Now, you (the students) are asked to investigate how the braking distance relates to speed just before braking. Advise the mayor about the consequences of changing maximum speed. Underpin your advice with representations like tables and graphs.



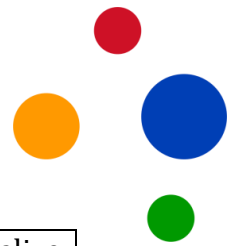
Consider a car braking in such a way that the speed decreases by 10 km/h every 0.4 seconds. You can use the tables below to organize calculations, observe, and then justify your answer as you best can.



Phase	Teacher's actions incl. instructions	Students' actions and reactions
<p>Devolution (didactical)</p> <p>10 minutes</p>	<p>The teacher divides students into groups of three or four.</p> <p>The teacher poses the problem to students. (S)He makes sure that students understand the assumption of a constant decreasing speed during braking and discusses the idea of small-time intervals where the movement can be approximated as the movement by the constant (average) speed.</p> <p>The teacher checks out that students understand the terms in the tables, the basic relationship between the speed, time and distance, how to convert km/h to m/s and the idea that 40 km/h could be replaced by other numbers.</p> <p>The teacher remarks to students they are free to use their own and different strategies. They are free to use any type of technology. Students are given a worksheet with the task. They are provided with a calculator (if students don't have their own), computer and graph paper.</p> <p>Students are told that they have 20 minutes to investigate how speed and distance are changing and to make some conclusions about how they are related.</p>	<p>Students listen, talk about their ideas and answer the questions.</p>
<p>Action (adidactical)</p> <p>20 minutes</p>	<p>The teacher circulates, observes students working without interfering.</p> <p>In the case that many groups start a new table for every new initial speed the teacher might ask for a</p>	<p>Students discuss in their group about strategies.</p> <p>They are completing tables using calculators or use ICT to graph points etc.</p>

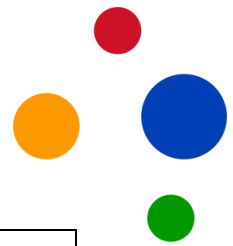


	<p>short plenary to ask how groups dealt with this issue. Probably, at least one of the groups realize that they can use previous calculations when trying to deduce the braking distance for other initial speeds and read from that table also the braking distance for lower initial speeds. This can be used as feedback for all other groups.</p>	<p>They talk about precision, choosing a different initial speed and similar issues. Members of the group might have different ideas and develop them individually.</p> <p>Students might use calculations, graphs or physics laws to come to conclusions:</p> <ul style="list-style-type: none"> - braking distance is not changing with a constant rate, - the relation between the initial speed and distance is not linear, - as the initial speed increases, braking distance also increases, but not proportionally. <p>Some students might notice that 2nd differences are (approximately) constant and use recursion method for calculations.</p>
<p>Formulation (didactical) 10 minutes</p>	<p>The teacher goes to each group and asks them to present shortly what they have found. (S)He might ask questions and discuss their ideas, particularly if they have stuck.</p> <p>The teacher asks groups with different strategies (within the group) to focus on one strategy that they will use for generalizing and presenting their ideas (due to lack of time).</p> <p>The teacher reminds students that the goal of the activity is to find out how the braking distance relates to speed just before braking to be able to do predictions and to give proper advice to the mayor. Therefore, students are asked to prepare advice to the mayor about the consequences of changing maximum speed and underpin their advice with representations like tables and graphs.</p>	<p>Students present their work shortly and ask questions.</p>



Action and formulation (adidactical) 20 minutes	The teacher is observing.	Students are trying to generalize their calculations and observations. Some of them might change the strategy for generalizing or approach to the problem. Students are preparing advice to the mayor.
Validation (didactical) 25 minutes	The teacher asks students to present and compare their strategies.	Students present their work, listen, ask questions and discuss other strategies and solutions.
Institutionalisation (didactical) 5 minutes	The teacher highlights the mathematical differences and similarities in the student's strategies, explains why some strategies will not provide proof but might be convincing from the graph and a formula that might be produced by technology, that the relationship is quadratic. The teacher introduces quadratic function.	Students listen and connect their solutions with a general quadratic function.

Possible ways for students to approach target knowledge	Students will fill the given table with data (v, d).										
		Time (seconds)	Change in speed during braking (km/h)	Average speed (km/h)	Average speed (m/s)	Time interval Δt	Distance traveled Δd (m)				
		$t = 0$ to $t = 0.4$	$v = 40$ to $v = 30$	35	$\frac{175}{18}$	0.4	$\frac{35}{9}$				
		$t = 0.4$ to $t = 0.8$	$v = 30$ to $v = 20$	25	$\frac{125}{18}$	0.4	$\frac{25}{9}$				
		$t = 0.8$ to $t = 1.2$	$v = 20$ to $v = 10$	15	$\frac{25}{6}$	0.4	$\frac{15}{9}$				
		$t = 1.2$ to $t = 1.6$	$v = 10$ to $v = 0$	5	$\frac{25}{18}$	0.4	$\frac{5}{9}$				
	Distance traveled after braking (m)						$\frac{80}{9}$				
	Speed just before braking (km/h)	30	40	50	60	70	80	90	100	110	
	Braking distance (m)	5	$\frac{80}{9}$	$\frac{125}{9}$	20	$\frac{245}{9}$	$\frac{320}{9}$	45	$\frac{500}{9}$	$\frac{605}{9}$	



Or with decimals, for instance:

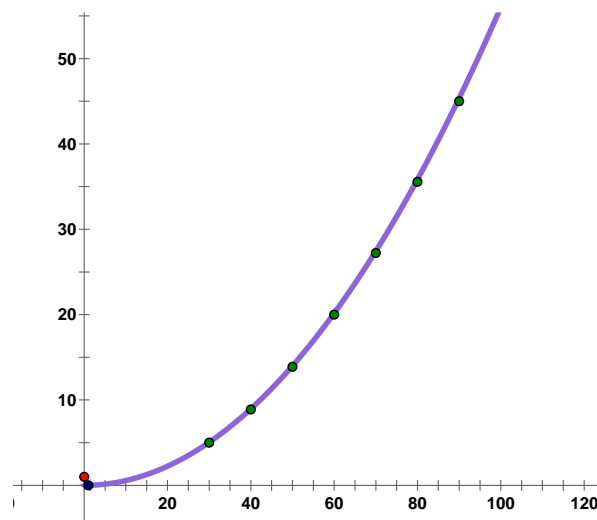
Speed just before braking (km/h)	30	40	50	60	70	80	90	100
Braking distance (m)	5	8.89	13.89	20	27.22	35.56	45	55.56

By looking at the data they can conclude:

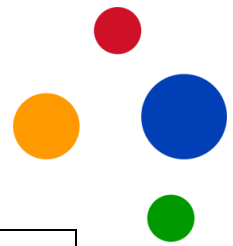
- The braking distance is longer when the speed is higher.
- The relationship between speed and braking distance is not linear ($\frac{\Delta d}{\Delta v}$ is not constant).
- If the speed doubles, the distance is increased four times. If the speed increases three times, the distance increases nine times.
- Students can draw points (v, d) and conclude that the relationship might be quadratic. They can write a quadratic function

$$d = av^2 + bv + c$$

and determine unknown coefficients a, b, c using data from the table and solving system of linear equations. They will get an approximation. This strategy will not provide proof that the relationship is quadratic.



- After the conclusion that the relationship might be quadratic, students can use ICT to find quadratic regression. They will get an approximation. This strategy will not provide proof that the relationship is quadratic.



- From the data in tables students can generalize:

$$d_{40} = 5 \cdot \frac{5}{18} \cdot 0.4 + 15 \cdot \frac{5}{18} \cdot 0.4 + 25 \cdot \frac{5}{18} \cdot 0.4 + 35 \cdot \frac{5}{18} \cdot 0.4$$

$$d_{40} = \frac{5}{9} (1 + 3 + 5 + 7) = \frac{5}{9} \cdot 16 = \frac{80}{9} \approx 8.89$$

$$d_{50} = d_{40} + 45 \cdot \frac{5}{18} \cdot 0.4$$

$$d_{50} = \frac{5}{9} (1 + 3 + 5 + 7 + 9) = \frac{5}{9} \cdot 25 = \frac{125}{9} \approx 13.89$$

$$d_{60} = d_{50} + 55 \cdot \frac{5}{18} \cdot 0.4$$

$$d_{60} = \frac{5}{9} (1 + 3 + 5 + 7 + 9 + 11) = \frac{5}{9} \cdot 36 = 20$$

$$d_{v_0} = \frac{5}{9} (1 + 3 + \dots + (2n - 1)) = \frac{5}{9} \cdot n^2$$

An important conclusion is that if we observe the braking distance, we look for the moment when the speed is equal to 0; so many times we will subtract 10 of v_0 until we get 0:

$$v_0 - 10n = 0 \Rightarrow n = \frac{v_0}{10}$$

$$d_{v_0} = \frac{5}{9} \cdot \left(\frac{v_0}{10}\right)^2 = \frac{1}{180} v_0^2 \approx 0.0056 v_0^2$$

In this formula we substitute v_0 in km/h and get the distance in metres.

- Students can use calculators and write data in tables as decimal numbers. The results will not be exact and it is not easy to recognize patterns.
- Students can use information that the speed decreases by 10 km/h every 0.4 seconds. They might calculate that the speed decreases by 25 km/h every second, or by 6.94 m/s every second, which means that acceleration is $a = 6.94 \text{ m/s}^2$. Then they use formulas from physics:

$$v = v_0 - at, d = v_0 t - \frac{a}{2} t^2.$$

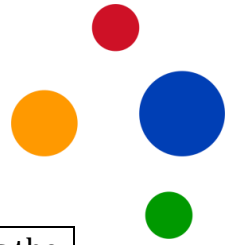
They use an important conclusion: if we observe the braking distance, we look for the moment when the speed is equal to zero. From the first formula ($v = 0$) they calculate time $t = \frac{v_0}{a}$ and substitute in the second to get

$$d = \frac{v_0^2}{2a} = \frac{9v_0^2}{125} = \frac{v_0^2}{13.8} = 0.072v_0^2.$$

In this formula we substitute v_0 in m/s to get the distance in metres.

- If students calculate acceleration in km/h² they will get: $a = 90000 \text{ km/h}^2$, substitute v_0 in km/h and get distance in kilometres

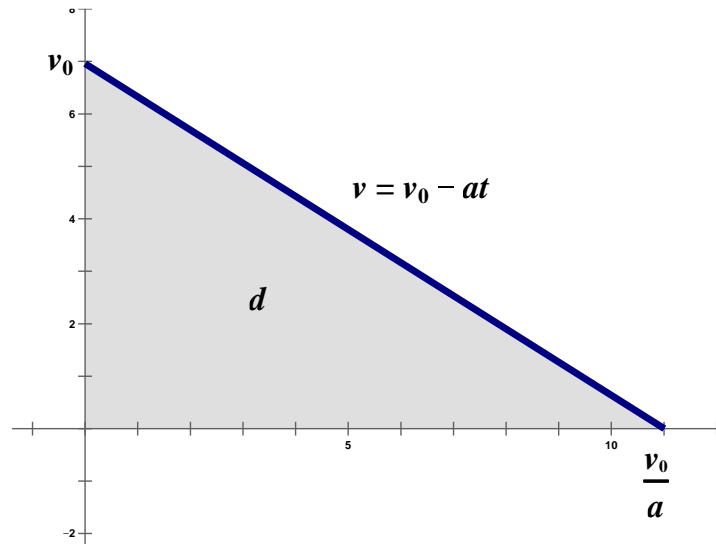
$$d = \frac{v_0^2}{180000}, \text{ or in metres } d = \frac{v_0^2}{180}.$$



- Students can draw a v - t graph and calculate the distance as the area under the graph:

$$d = \frac{1}{2} \cdot \frac{v_0}{a} \cdot v_0 = \frac{v_0^2}{2a} = 0.072v_0^2.$$

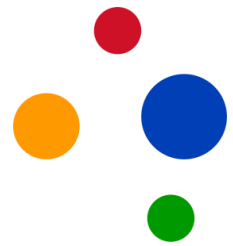
In this formula we substitute v_0 in m/s.





	Time (seconds)	Change in speed during braking (km/h)	Average speed (km/h)	Average speed (m/s)	Time interval Δt (s)	Distance traveled Δd (m)
	$t = 0$ to $t = 0.4$	$v = 40$ to $v = 30$	35			
Distance traveled after braking (m)						

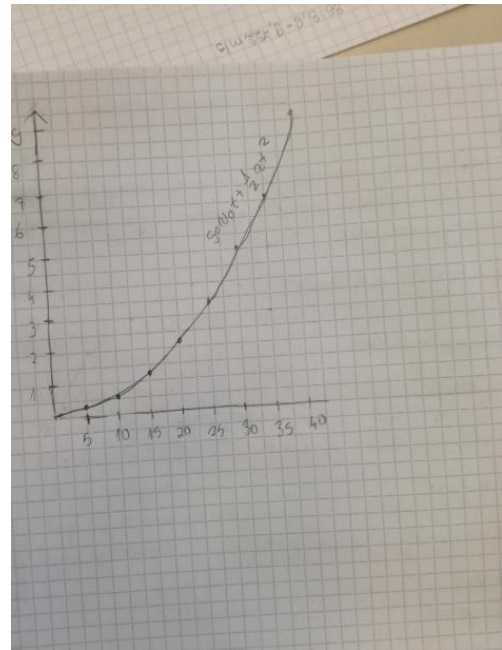
Speed just before braking (km/h)	40							
Braking distance (m)								



Explanation of materials

In the beginning, the students will get the tables they need to fill out. The aim is to encourage them to observe 0.4 second intervals, average speeds at these intervals (in km/h and m/s) as well as the estimated distance in that time interval. They will decide to count the parts of the distance until the speed becomes zero. The number of rows in the table will, therefore, be changed and the students will determine how many rows of tables they need. In the second table, they will, apart from the proposed speed of 40 km/h, choose other speeds to observe by themselves. Students will use tables in the action phase. When completing the table for various initial speeds takes too much time, sharing work could be advised. Students should realize that they could use previous calculations when trying to deduce the braking distance for other initial speeds and read from that table also the braking distance for lower initial speeds. In the case that many groups start a new table for every new initial speed somewhere during the 20 minutes didactical phase the teacher might ask for a short plenary to ask how groups dealt with this issue. Probably, at least one of the groups found a solution for that and this could be used as feedback for all other groups.

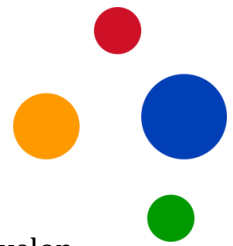
To draw graphs, students can use millimetre paper, square paper, or computer. Students, who successfully conclude about the sum of $1 + 3 + \dots + (2n - 1)$, can be offered cubes as a visual aid to make a proof without words.



Variations based on didactic variables

In the implementation of the scenario, the envisioned didactic and didactical phases should be kept and the action phase should be done didactically. It is important to conduct a validation phase in which students will evaluate the presented solutions. Some parts of the scenario can be changed. In this chapter, we will list the didactic variables or parts of the scenario that can be changed as well as teacher's interventions that should be implemented in some situations.

Didactic environment: The problem can be presented in different ways. The teacher can talk about the content of the problem, can use presentation or video. The speed of 40 km/h is chosen arbitrarily and can be replaced by another but due to the importance of noticing the regularity, we recommend it to be a multiplier of 10. Speed reduction data for 10 km/h every 0.4 seconds is selected so that it is realistic, while at the same time it allows the students to notice the regularity, so it should not be altered. Tables are provided to help organize the data, but should not be considered as a requirement since the students can reach general conclusions without calculating the braking distance for specific speeds. The tables are only partially set, so students who use them must determine how many rows should they do the calculation for in the first table and which speeds are to be included in the other.



The teacher should avoid giving precise tables in order to enable students to develop research skills. Students can be encouraged to use ICT for drawing, displaying, and computing, but the scenario can be implemented without the use of ICT. The teacher can create materials for the use of ICT. If a teacher creates material, one should keep in mind that ICT only helps with the calculation and the display of the results, but it does not offer conclusions.

The duration of individual phases can be tailored to students, but deviations should not be great.

If during the *adidactic phase* the groups have not found a general formula that connects the braking distance to the initial speed, the teacher may ask the following questions:

- Can you notice some regularity between the braking distance values obtained?
- Can the resulting values be graphically displayed? Can you link the graphical and algebraic displays?
- What is the speed of the vehicle when it stops?
- If you have determined the braking distance for different initial speeds while you were filling out the tables, can you do the same, but for a general v rather than a specific one?
- Which formulas from physics can be useful?
- How can technology help you find formulas or relations?

The teacher does not need to teach each group individually. Furthermore, it is not necessary to stay with the group until they have answered the questions asked. Consider these issues as a small *devolution* to a limited problem and allow students to continue adidactically with the *action phase* and *formulation*. The teacher should not support discussion on further issues nor should they suggest any answers.

For groups that have a problem proving the formula: $1 + 3 + 5 + \dots + (2n - 1) = n^2$ the teacher can say:

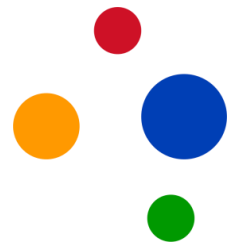
- present each number in the sum by a dot and observe;
- mark the sum with S and write it twice, but in a different order: from the first to the last and from the last to the first.

It is not necessary for the students to prove the formula in the adidactic phase. It is worthwhile to find that the sum observed is equal to n^2 and the proof can then be conducted didactically at in the validation phase.

The students will choose the starting speeds themselves. They will probably choose speeds of 50, 60, 70 ... km/h. If they choose speeds that are not multiples of 10, they will have more trouble determining the moment when the speed becomes 0 because it will not be a multiple of 0.4 s. In this case, it will be more difficult to determine the regularity. The selection of observed speeds can be discussed at the validation phase.

Students can display the braking distance as a fraction or a decimal number. Decimal numbers will be only an approximation so some regularities will not be noticeable. The difference between the two displays can also be discussed in the validation phase.

During the *institutionalization phase*, it is important that the majority (if not all) strategies that have been talked about in the classroom are commented and linked to each other.

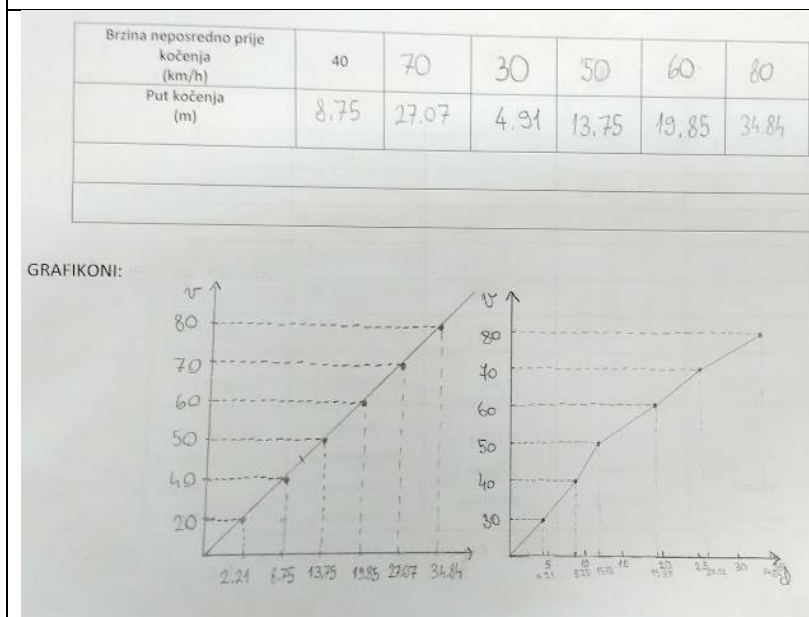


Observations from practice

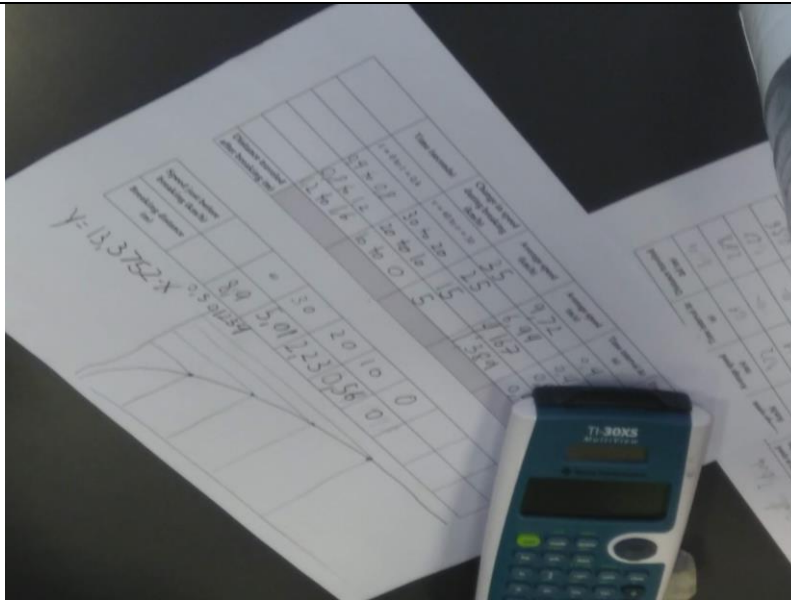
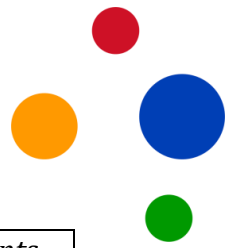
In some groups, students made mistakes in calculations and get wrong braking distance for some initial speeds. In the first formulation phase, the teacher can ask students from different groups to compare their results and to correct the wrong ones.

$8.88 = 0.40 + b$
 $8.88 = 0.30 + b \rightarrow b = -0.30 + 5$
 $8.88 = 0.40 - 0.30 + 5$
 $3.88 = 1.0a$
 $a = 0.355$
 $8.88 = 0.355 \cdot 40 + b$
 $8.88 = 14.2 + b$
 $b = 6.64$
 $y = 0.388x - 6.64$

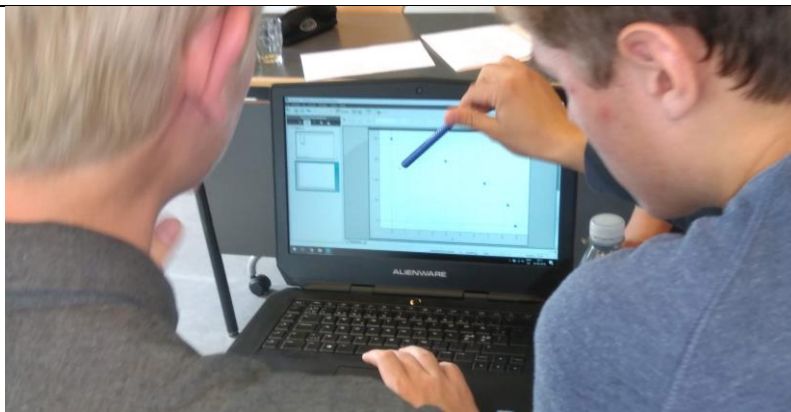
Some of the students assumed that the relationship between speed and braking distance is linear. They used data from the tables to determine the linear function.



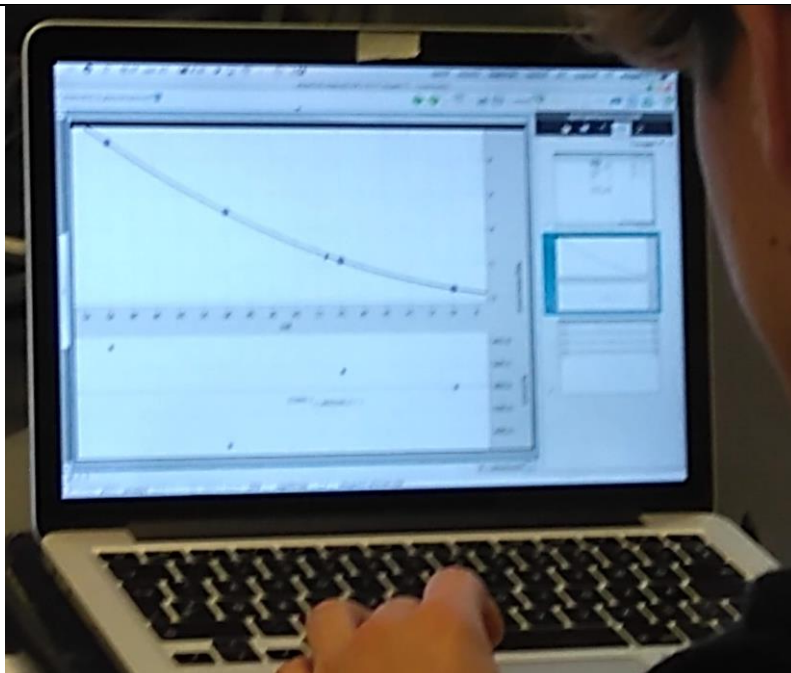
In some groups, students tried to draw points so they lay on a line, or as a graph of the piecewise linear function. In that case, the teacher can ask students to explain why they think the relationship is linear, do they know some properties of the linear function and can they find those properties in data. Students should realize that the relationship is not linear because the difference quotient is not constant.



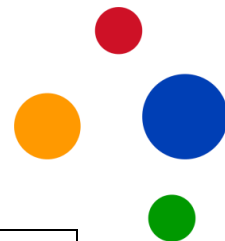
In some groups, students put distance on the horizontal axis.



Some students used ICT.



After concluding that the relationship might be quadratic, students found the quadratic regression.



$x = v_0 t + \frac{1}{2} a t^2$
 $x = 11,11 \cdot 0,4 - \frac{1}{2} \cdot 2,778 \cdot 0,16$
 $x = 4,4444 - 0,2222$
 $x = 4,2222 \text{ m}$

$v = \frac{a \cdot t}{2} = \frac{-2,778 \cdot 0,4}{2} = -0,5556$
 $v = 0,5556$

$a = \frac{v - v_0}{t} = \frac{-0,5556 - 11,11}{0,4} = -27,775$

Vrijeme (sekunde)	Promjena brzine za vrijeme kočenja (km/h)	Prosječna brzina (km/h)	Prosječna brzina (m/s)	Vremenski interval Δt	Pređeni put Δd (m)
$t = 0$ do $t = 0,4$	$v = 40$ do $v = 30$	34,92	9,7	0,4	3,88
0,4 - 0,8	30 - 20	25,02	6,95	0,4	2,78
0,8 - 1,2	20 - 10	15,03	4,175	0,4	1,67
1,2 - 1,6	10 - 0	5,04	1,4	0,4	0,56

Put kočenja (m): 8,9

Brzina neposredno prije kočenja (km/h)	40	30	20	10	0	100	110	
Put kočenja (m)	0,5	13,58	19,99	22,99	25,53	49,96	55,51	67,92

$\frac{5}{9} (4 + 3,5777; 2,7; 1,9; 1,1; 0,5)$
 $\frac{5}{9} (1 + 3,5 + 7 + 9 + 11 + 13 + 15)$
 $\frac{5}{9} (1 + 3 + 5 + 7)$
 $\frac{5}{9} (16)$

$x = 8,33 \cdot 0,4 - \frac{1}{2} \cdot 6,94 \cdot 0,16$
 $3,332 - 0,555$
 $2,777$

$x = \frac{a}{t}$
 $x = \frac{v_0 t - \frac{1}{2} a t^2}{t}$
 $11,11 - \frac{1}{2} \cdot 2,778 \cdot 0,4$
 $11,11 - 0,5556$
 $10,5544$

In some groups, students used fractions and came to the sum of consecutive odd numbers.

Udregning:

$25 \text{ km/t} \cdot 0,2777 \text{ m/s} = 6,944 \text{ m/s}$
 $6,944 \text{ m/s} \cdot 0,4 = 2,777 \text{ m}$

$15 \text{ km/t} \cdot 0,2777 \text{ m/s} = 4,166 \text{ m/s}$
 $4,166 \text{ m/s} \cdot 0,4 = 1,666$

Generalisering:

Gennemsnittsfart målt i km \cdot (m/s \cdot tidsinterval målt i sek)

$\frac{m}{s} \cdot s = \frac{m \cdot s}{s} = \frac{1000 \cdot 0,4}{3600} = 0,111 \text{ m}$

Bremselængde

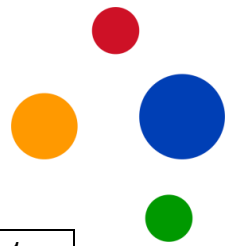
$y_1 = x_1 \cdot 0,111$
 $y_2 = x_2 \cdot 0,111$
 $y_3 = x_3 \cdot 0,111$

Gennemsnittsfart = x
Kort afstand = y (målt i m)

Således bremselængde

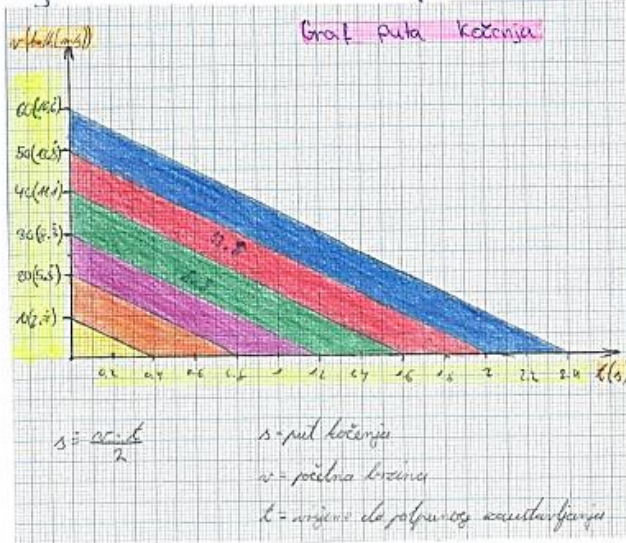
$y_1 + y_2 + y_3 + \dots + y_{10} = x_1 \cdot 0,111 + \dots + x_{10} \cdot 0,111$
 $= (x_1 + x_2 + \dots + x_{10}) \cdot 0,111$

Students came to the sums of some other sequences. In this case, the teacher should finish their work in the institutionalisation phase.



Poštovana gradonačelnice!

Ovim pismom Vam iznosimo prijedlog o smanjenju ograničenja brzine u Hallerovoj aleji. Prijedlog se zasniva na podacima iz grafa duljine puta kočenja. Željeli bi da se naš prijedlog ostvari zbog izloženosti djece opasnostima koje dolaze od strane neopreznih vozača.



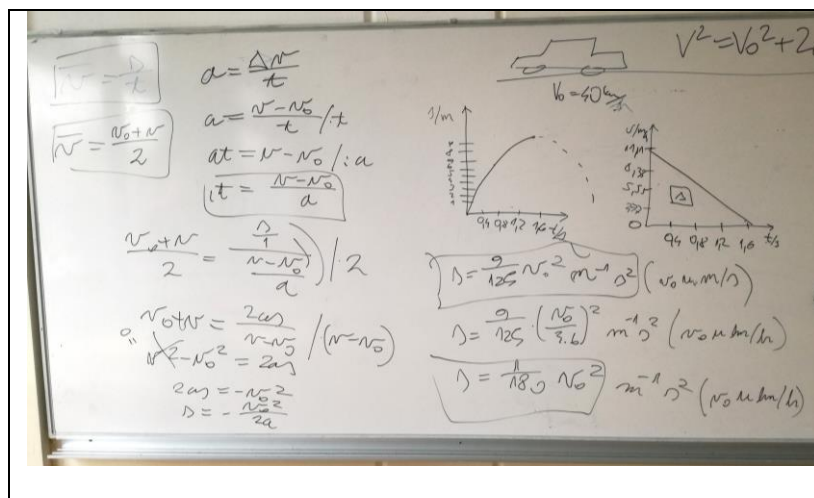
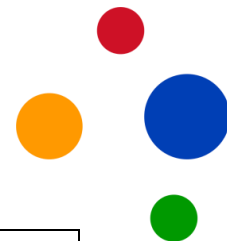
Sa poštovanjem

Vijeće roditelja

Some students draw v-t graphs for different initial speeds and calculate braking distance as the area under the graph. In this case, the teacher should continue with this idea in the institutionalisation phase and draw the v-t graph for a general v.

$v = 50 \text{ to } 40 \rightarrow 45$
 $v = 60 \text{ to } 50 \rightarrow 55$
 $y = 0,16 \cdot t^2 \quad s = vt$
 $v^2 = v_0^2 - 2 \cdot a \cdot s$
 $0 = v_0^2 - 2 \cdot a \cdot s$
 $2ay = v^2$
 $s = \frac{v_0^2}{2 \cdot a}$
 $s = \frac{v_0^2}{13,88}$
 $v = v_0 - at$
 $a = \frac{v_0}{t}$
 $a = 6,94$

Some students combine formulas $a = \frac{\Delta v}{t}$, $\bar{v} = \frac{s}{t}$ and $\bar{v} = \frac{v_0 + v}{2}$, get the formula $v^2 - v_0^2 = 2as$ (1). Since speed v for the braking distance is equal to 0 it would follow that the $s = -\frac{v_0^2}{2a}$ where acceleration a is taken with a negative sign.



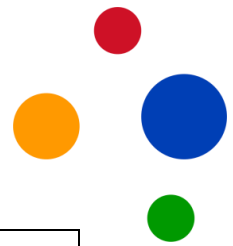
Formula (1) can be obtained by eliminating time t in formulas $v = v_0 + at$ and $s = v_0 t + \frac{a}{2} t^2$ or the students might already know it from physics.

Using the formula (1), students will get the quadratic dependence of the braking distance and the starting speed of the car without calculating specific values. The groups that handle the problem in this way will likely be faster than the other groups and can be offered to go through tables all the same, and to pay attention to data required in tables (e.g. why the average speed is a relevant data?), assigned the task of displaying the relationship of the braking distance and the initial speed graphically, preparing an explanation of the importance of the speed limit near the schools, or investigating the dependence of the *stopping distance* and the initial speed (see Suggestion for further problems 1.)

Conclusion:

We can see that some students draw conclusions only by looking at the numbers while some are trying to describe how the braking distance is related to speed just before braking. See the table below.

Numbers		Relationships	
Students calculate the braking distance for a few speeds, for instance, 40 km/h and 70 km/h, and conclude that the braking distance is too long if the car is driving 70 km/h so they recommend speed limit 40 km/h.		Students conclude looking at numbers that the relationship is not linear (for higher speeds change in the speed results in a more dramatic change in distance).	
Students fill the table with more data, for instance		Students looking at the numbers in the table conclude that the relationship could be quadratic because if we double the speed the distance will increase four times (looking at 30 km/h and 60 km/h).	
Speed (km/h)	30 40 50 60 70 80 90		
Braking distance (m)	5 8.89 13.89 20 27.22 35.56 45		
and make recommendations by looking at the numbers.			
Students plot points and make a recommendation by looking at them.		Students plot points and conclude that the relationship could be quadratic and draw the graph.	



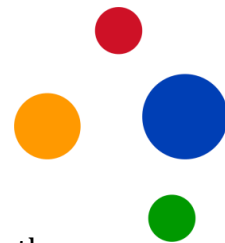
	<p>Students conclude that the relationship is quadratic and use technology to find quadratic regression.</p>
	<p>Students conclude that the relationship is quadratic and use the data from the tables to find coefficients for a quadratic function.</p>
	<p>Students use arguments to prove that the relationship is quadratic (see <i>Possible ways for students to approach target knowledge</i>).</p>

Although computing and collecting data is important in this scenario, students should feel the need to explore further and try to determine the relationship. If all the students are satisfied with the answers obtained only by observing the numbers, the teacher could discuss what “determining the dependency” means in the institutionalisation phase.

Evaluation tools

At the end of the class or at the beginning of the next one, the students can be given several tasks:

- Two cars are passing down the street. The speed of one is three times the speed of another. Will the braking distance of the faster vehicle be three times bigger? Explain the answer.
Answer: No. The braking distance is quadratically dependent on the speed immediately before braking. That is why the braking distance of the faster vehicle will be nine times bigger.
- The vehicle moves at a speed of 80 km/h. To what speed should the speed be reduced if we want the braking distance to be twice as small?
Answer: The speed should be reduced by four times, or reduced to 20 km/h.



3. The braking distance is affected by the initial speed as well as the weather conditions on the road. One day a measurement was made and it was found that the vehicle that moved at a speed of 40 km/h stopped after 10 m. After how many metres will the vehicle, moving in the same conditions at a speed of 70 km/h, stop?

Answer: The braking distance is quadratically dependent on the initial speed so we assume that this dependence can be written as $d(v_0) = kv_0^2$. In this task, it would mean that $d(40) = 10$ so $k = \frac{d}{v_0^2} = \frac{1}{160}$. A vehicle moving at a speed of 70 km/h will stop after $d(70) = \frac{70^2}{160} = 30.625 \approx 30.6$ m.

Suggestion for further problems

1. The stopping distance of the vehicle consists of two parts: the reaction distance and the braking distance.

The distance that a vehicle passes from the moment when the driver sees the need for braking to the moment when he starts to brake is called the reaction distance. The driver's reaction time is 1 s and can be extended due to the characteristic of the driver, illness, and drug and alcohol fatigue. The reaction time of the driver under the influence of alcohol (0.5 g/l of alcohol in the blood) is 1.5 s. We assume that in the period of the reaction time the vehicle drives at a constant speed.

The distance that a vehicle passes from the moment the driver starts to brake until it stops is called the braking distance. The braking distance depends mostly on speed just before braking (so-called initial speed) and road conditions, and may also depend on the condition of the vehicle. If neglected the condition of the vehicle the braking distance is calculated by the formula $s = \frac{v^2}{254\mu}$ where v is the speed in kilometres per hour, and μ the coefficient of friction, which depends on the road conditions:

Friction coefficient μ	Dry road	Wet road
asphalt new	0.7 – 0.8	0.5 – 0.6
asphalt old, dirty	0.6 – 0.7	0.25 – 0.45
pebble, small stone	0.6 – 0.7	0.3 – 0.5
snowfall	0.2 – 0.4	
ice	0.05 – 0.1	

- a) Explore how the reaction distance depends on the speed just before braking for the driver with the reaction time of 1 s and for the driver with a reaction time of 1.5 s.
- b) Find out how the braking distance depends on the speed just before braking for dry and wet asphalt road and road covered in snow.
- c) Find out how the stopping distance depends on the speed just before braking for different reaction times and different conditions on the road.
2. How many diagonals have a quadrilateral, pentagon, hexagon, and n-gon?



3. How many pieces of pizza can you get if you cut it two, three, four, n times?
4. The equilateral triangle of the side length n cm is divided into equilateral triangle sides of 1 cm long. How many are there?
5. Describe the mathematical path of the ball at free throws in basketball.

Rationale and RME perspectives on the scenario

Relevance and applicability

This knowledge relates to daily experiences with vehicle movement and braking. Students become aware that the speed immediately before braking affects the braking distance. Knowledge and skills related to the topic of quadratic dependences are present in many areas.

Inquiry skills

The inquiry is included in all of the phases. Students should be accustomed to inquiry and more often put into situations where they will work in this way. Thus, while developing mathematical competence they also develop inquiry skills. During the scenario implementation students will generate examples, experiment systematically, organize data, formulate hypotheses, find and justify formulas, collaborate and communicate. Inquiry skills should be included in the feedback in the phase of validation and institutionalization.

Potential for a sequence of lessons

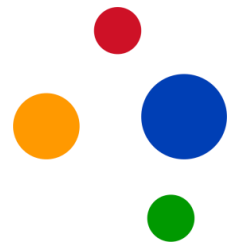
The scenario can be part of a larger set of lessons about quadratic dependencies and the properties of quadratic sequences and quadratic functions.

- *Preliminary knowledge:* For the chapter on quadratic dependencies, we expect students to understand the concept of a function, in particular, linear function, the concept of an arithmetic sequence and its properties.
- *Introduction:* A context of a braking car can be used as a wide-open problem involving the module.

Rationale for the scenario

- *Horizontal mathematizing:* The mathematical language is introduced to discuss the situation. The students make the first informal model of the situation - the scenario of the braking distance, quadratic dependency is introduced.
- *Vertical mathematizing:* math involved in the problem is further developed. The model is made more compact, more general. Students investigate patterns in the numbers and their sums.


Students study quadratic sequences and the characterization of quadratic sequences: the first differences are linear and the second ones are constant. Furthermore, sums of terms in linear (arithmetic) sequences are quadratic sequences. Generalization - quadratic function, the first derivation is linear and the second one is constant. Furthermore, integral of a linear function is quadratic.



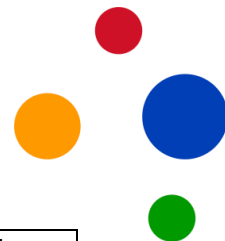
MERIA Module “Conflict lines – introduction”

Partitioning of a plane by perpendicular bisectors

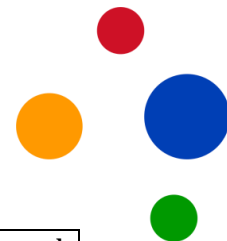
The teaching scenario

Target knowledge	The partitioning of a plane by perpendicular bisectors of pairs of given points.
Broader goals	<p>Construction of a perpendicular bisector. Understanding the characterization of a perpendicular bisector as the collection of points that have equal distance to two given points. Characteristics of bisectors and their points of intersection in triangles and quadrilaterals, and characteristics of points in regions determined by perpendicular bisectors. The ability to operate with the notation $d(P,X) < d(P,Y)$.</p> <p>Inquiry skills: experimenting and drawing systematically to create areas or borders of areas that are determined by (distances to) given points. Presenting findings clearly by making decisions which lines to emphasize.</p> <p>Interdisciplinary skills: students can connect territorial problems or conflicts (geography) to geometrical ways of representing and solving these conflicts. Other problems may be used to discuss the application to robot navigation.</p>
Prerequisite mathematical knowledge	Pythagoras and triangle inequality (in particular for the proof).
Grade	Age 15 - 16, grade 9 - 10 (whenever the perpendicular bisector is introduced)
Time	40 minutes, with applet 70 minutes
Required material	<p>Worksheets, paper, ICT and MERIA applet in GeoGebra: https://meria-project.eu/applet/voronoi/voronoi.html Alternative sites: http://alexbeutel.com/webgl/voronoi.html https://www.desmos.com/calculator/ejatebvup4</p>
Problem:	<p>Given a collection of water wells in a desert. Students are asked to colour areas in the desert in such a way that for each possible point in a coloured area the corresponding well should be the one that is the closest to that point.²</p> 

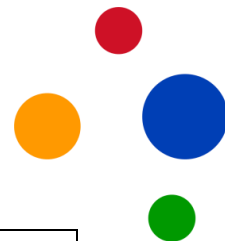
² The problem and the map of the desert was introduced in the book Geometry with Applications and Proofs, Voronoi Diagrams by A. Goddijn, M. Kindt, W. Reuter



Phase	Teacher's actions incl. instructions	Students' actions and reactions
Devolution 1 (didactical) 5 minutes	Introduce the notion of a conflict line in the classroom: Suppose two students (X and Y) have some sweets and you are asked to go for a sweet to the one who is closest to you. The teacher selects the two students and asks: who is closer to student X and who to student Y, and finally let students raise hands that have difficulty in deciding ...	Students participate by raising hands and feel being a point in a plane and deciding themselves whether they are closer or not to one of the two given points. Moreover, they see how others decide.
Institutionalisation (didactical) 2 minutes	The teacher summarizes the main finding: The problem is to identify points with 'same distance' and the challenge is to find some kind of procedure for being sure and precise about the collection of points with that characteristic. Consensus is established on notation (e.g. $d(A,C) < d(B,C)$ for point C being closer to A than to B). This will be elaborated in the next step (students will get the opportunity to work with notation and distance-related reasoning).	Students listen and are able to connect the institutionalized reasoning and notation to their own work.
Devolution 2 (didactical) 3 minutes	The teacher sets a new problem: Locate yourself somewhere in the desert (provide a worksheet to students). Find the well that is the closest to you. Find all positions from which you would also go to that well. Finally, divide the map into regions around wells, such that for each well all points in the corresponding region are closest to that particular well.	Students listen.



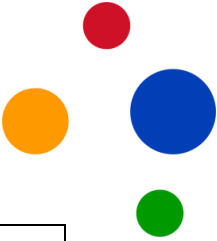
<p>Action (adidactical)</p> <p>15 minutes</p>	<p>Teacher circulates in the classroom.</p>	<p>After plotting the position and detecting the closest well, groups start to construct the region with all such points – the closest well paired with others, one by one. To divide into regions students discover that they need some kind of strategy (proof) because soon, with more points, things become complicated.</p>
<p>Formulation (adidactical)</p> <p>5 minutes</p>	<p>Teacher circulates in the classroom to identify what different ideas the students recall and use and announces presentations.</p>	<p>Students discuss in groups what they did, what the set of points with the requested property is and how to write it down.</p>
<p>Validation (didactical and adidactical)</p> <p>5 minutes</p>	<p>Teacher asks some groups to present what they did so far (if possible, at least a group that uses equidistant circles and a group that started drawing bisectors).</p>	<p>Students present.</p>
<p>Institutionalisation (didactical)</p> <p>5 minutes</p>	<p>Teacher highlights the fundamental theorem underlying what they did: $d(A,P) = d(B,P)$ if and only if P is on the perpendicular bisector. Voronoi diagrams are constructed with perpendicular bisectors, so these are the basis of algorithms to construct those diagrams. In addition, definitions of bisectors can be discussed: “the collection of points with equal distance to points A and B”, and “the line through midpoint and perpendicular to AB”. An optional part of the scenario: Can you prove the theorem? (not all students feel the need for proof and see it as a challenge)</p>	<p>Students understand the introduced notation as it refers to their activity, e.g. $d(A,P)=d(B,P)$ defines a line of points P (so-called “conflict line” for points A and B). $d(A,P)<d(B,P)$ defines a region (so-called “safe region”), and they understand the mathematical problem as it also emerged in their activity.</p>
<p>Devolution 3 (optional)</p> <p>5 minutes</p>	<p>ICT can be used to draw Voronoi diagrams. The teacher demonstrates what an ICT program can do (after entering two points). Recall/discover there is a line with equidistant points, and the</p>	<p>Students listen and watch the software drawing Voronoi diagrams automatically. They become interested and challenged to use it by themselves and investigate what happens when playing around.</p>



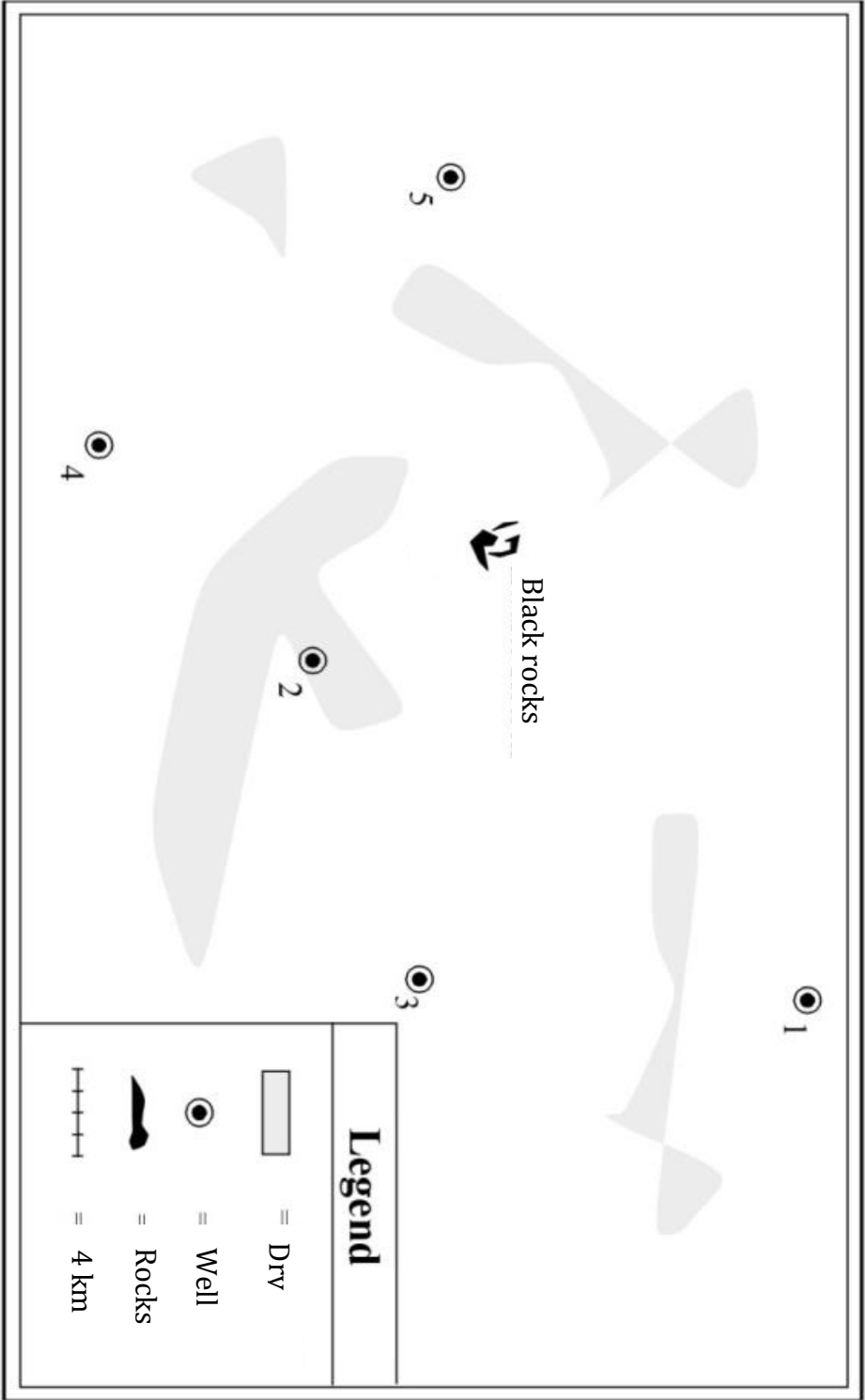
	<p>plane is divided into 2 regions. Then continue with 3 points and find/discover there exists an equidistant point to all 3 points. Use for instance: https://meria-project.eu/applet/voronoi/voronoi.html. Revisit your original problem-situations and explore what happens in specific cases. Play around and find a nice surprising pattern with a set of structured points or, for instance, explore what patterns you can get with 4 points moving around.</p>	
<p>Action (adidactical) 10 minutes</p>	<p>Teacher circulates in the classroom, challenging students to experiment systematically, only supporting them when they have problems with operating the software. When many have a similar problem, deal with that problem plenary (e.g. note it on a visible spot).</p>	<p>Students construct their original problem in the software, find the solution and compare it with their original drawing. They also explore what happens in other cases with regularly and/or irregularly distributed points.</p>
<p>Formulation (adidactical) 5 minutes</p>	<p>Teacher asks them to prepare a presentation of their (most surprising) findings and challenges them to find justifications for specific patterns (e.g., when does a 4-point Voronoi diagram have one 4-region point?).</p>	<p>Students prepare two screenshots, one of the solutions of the original problem and one of their nice pattern (and how they constructed it). They try to formulate justifications for their findings using the circle-tool and the formal distance-notation and theorems like Thales, Pythagoras ...</p>
<p>Validation (didactical and adidactical) 5 minutes</p>	<p>The presentations help to validate what happens in these diagrams, to get familiar with the formal notation and to use geometrical reasoning in different partitioning situations.</p>	<p>Students see the connection between the validations and the formulations of their findings.</p>
<p>Institutionalization (didactical) 5 minutes</p>	<p>General conclusions of the concept of Voronoi diagrams consisting of perpendicular bisectors, and some illustrative cases and patterns in these diagrams.</p>	<p>Students realize how institutionalized learning goals are connected to their initial explorations in the desert-context and have acquired these learning goals.</p>

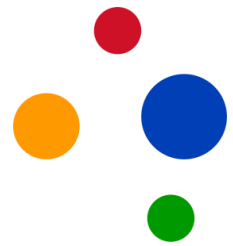


<p>Possible ways for students to realize target knowledge</p>	<ul style="list-style-type: none">• Some students will start sketching lines between the given points that have more or less curved pieces and no clear intersection-points where three (or four) lines meet.• Some students will draw circles or divide the areas with curved lines. These students need to realize that curved lines are impossible and that drawing circles is helpful for finding points with the same distance to a well or center, but not for finding borders (although they can be used for that).• Some students might immediately know what to do and start drawing bisectors. For them, the crucial point is to discuss what happens in areas where bisectors meet. Do they meet at one meet point?
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Worksheet





Explanation of materials

A map of a desert with wells is provided. The teacher is expected to prepare the map of the desert for each group. Some of the graphical elements in the map, like the black rocks, show that part of the mathematization from a real desert into a map is done, but not all. In case you do not want to bother students with these elements, you can decide to erase or ignore them. An applet is provided for exploring several situations. Check in advance whether the applet is operational, how to use it as a demonstration tool, and decide how and when to use it with what task/question for the students.

Variations based on didactic variables

Starter: Check how to organize the introductory problem. When you hand out, create critical situations during which some students need to hesitate... After devolution 1 the teacher can decide whether or not to introduce a formal notation for the distance dependent on the prerequisites of the students.

Milieu: We create a problem situation with the desert and the wells. You could create an alternative by using another context or different number and position of the points. Other ways of phrasing the problem: describe a strategy by which you can decide for as many positions in the desert as possible to which well to go. The nice aspect of this problem is that the milieu provides the criteria for validating student work. The winning strategy, a strategy for deciding for all points where to go to, except the ones on the conflict lines, is in line with the target knowledge aimed at.



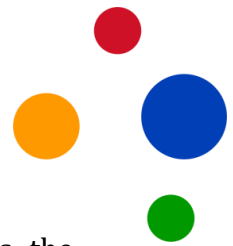
Dividing the map such that it is possible to decide for each point in Slovenia which helicopter deck is closest. (Nadja Marušič, Slovenia)

The length of the phases can be adapted to students' work.

During the first action phase:

In case almost all students use perpendicular bisectors and the action phase is quickly finished, put more emphasis on action-elements in proof or exploration with the applet:

- Considering the proof, you could ask the students to prove the perpendicular bisector theorem and/or equality between a point on Voronoi edge and a point on bisector.
- With the applet you can let them draw a variety of (interesting patterns), but this might end up in a difficulty for validating and institutionalizing some mathematical ideas/findings. We suggest providing students in such a case with clear questions like: Explore possible situations with 4 points (how many different patterns can you find? When do all bisectors meet? ...). Challenge them to provide criteria and/or proofs.



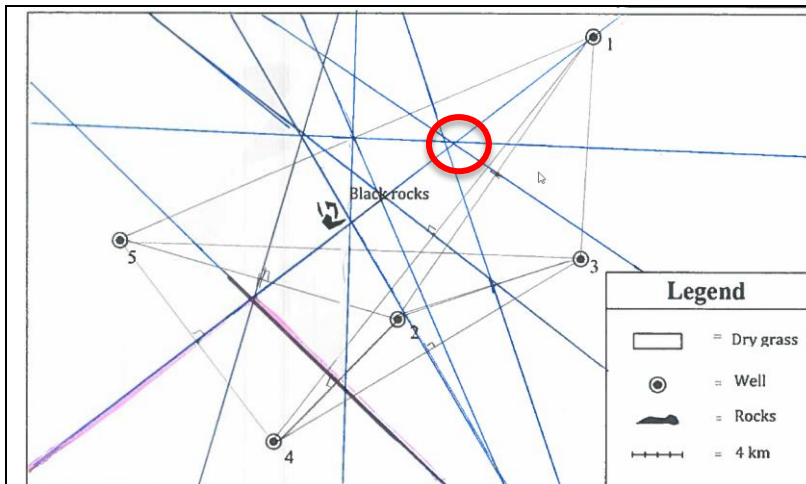
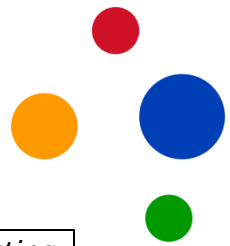
If some of the students do not have the prerequisites needed for making progress, the teacher should pose questions such as: How is this problem connected to the introduction (devolution 1)? How does the dividing issue between two points connect to the introduction? How can you decide between two given points? Why? The proposed questions should only be posed to groups or individuals if most other students seem to have the required knowledge. The teacher should not lecture each group separately. Further, it is *not needed* to stay with the group until they reach an answer to such a question. Regard it as a minor devolution of a limited problem and let the students act, formulate and validate. Do not support with further questions or hint the answer. If the majority of the class needs to consider those questions, the phase should be shortened and they should be given in plenum; such a need is usually a sign that the initial problem was too difficult or not posed clearly, which one should strive to avoid.

In case almost none of the groups use perpendicular bisectors, discuss the above-suggested questions plenary (instead of going from group to group).

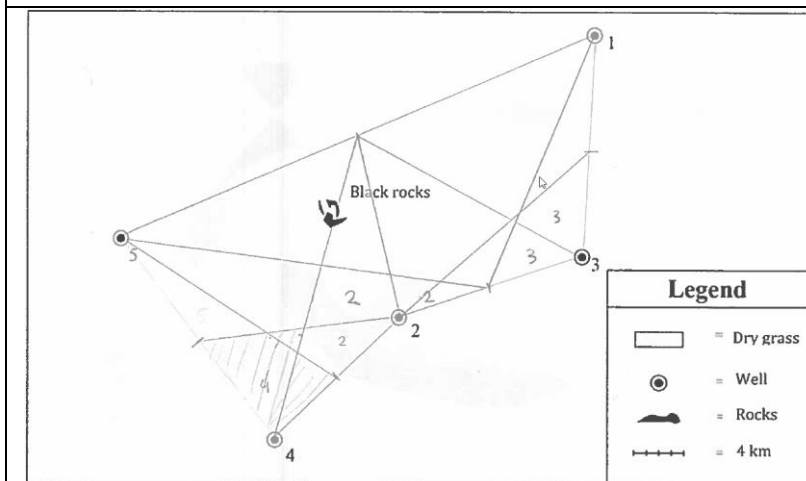
Observations from practice

During the action phase, students formulated the following approaches:

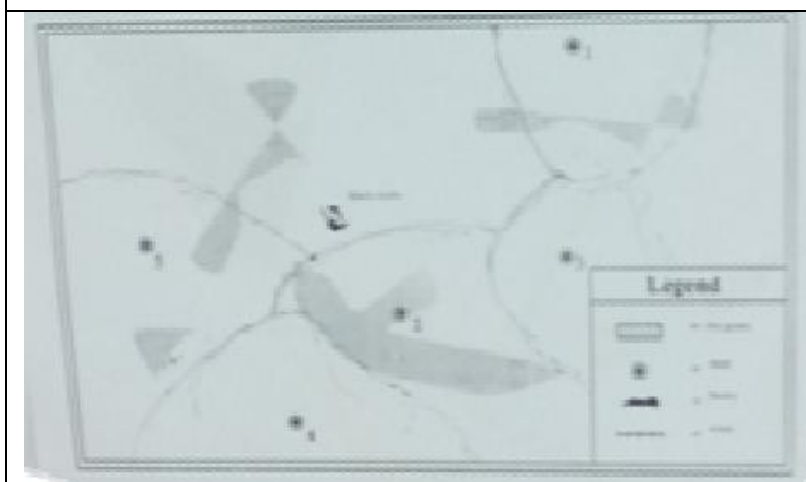
	<p><i>Students start with drawing circles around wells. Next, they connect points and proceed with perpendicular bisectors. It is difficult to see which regions belong to wells and it is not fully clear what happens where bisectors (seem to) meet.</i></p>
	<p><i>These students also start with circles, draw dotted connecting lines and find midpoints. They seem to proceed with circles touching midpoints, and they draw (some) perpendicular bisectors.</i></p>



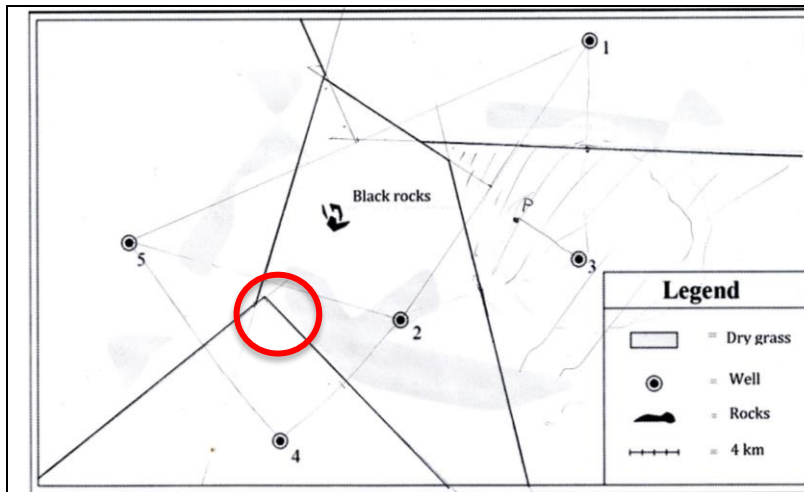
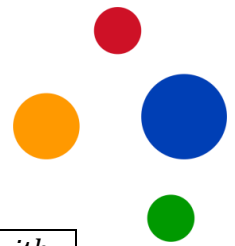
Students draw connecting lines and proceed with all perpendicular bisectors. It is difficult to see regions belonging to wells, and not clear what happens where bisectors meet (e.g., what happens in the red circle?).



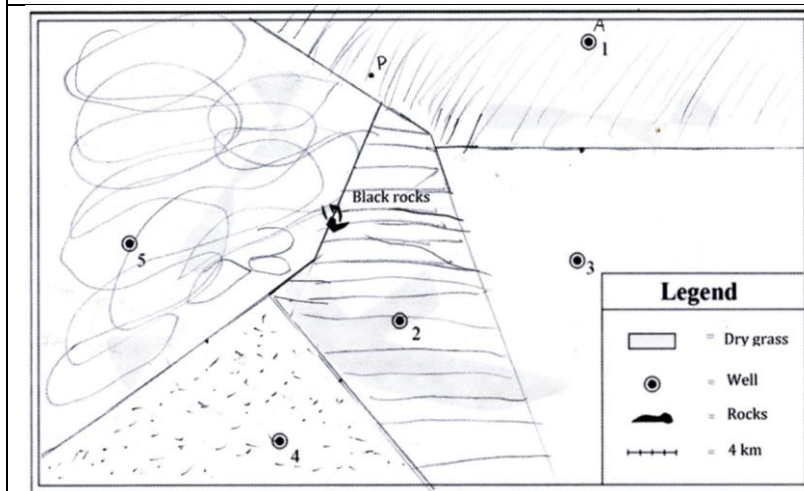
Students draw connecting lines and midpoints. Next, they connect midpoints and assign regions. The assigned regions are (almost all) correct, but not everything can be decided. No perpendicular bisectors!



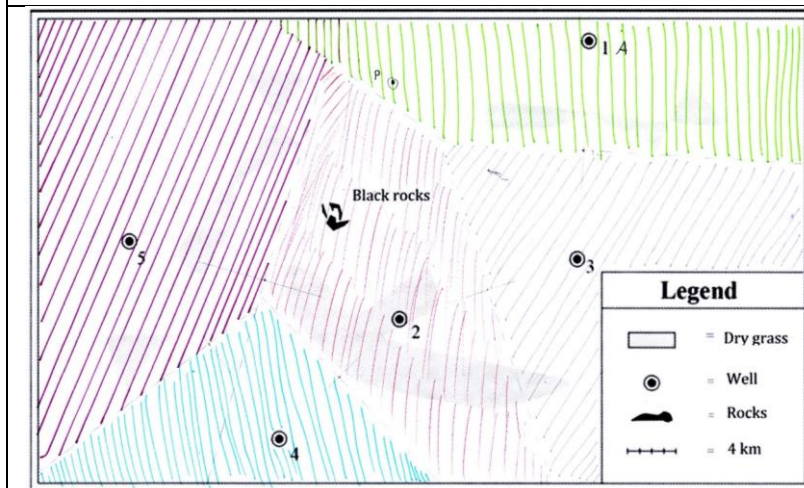
Students draw regions not using any mathematical strategy with curved lines.



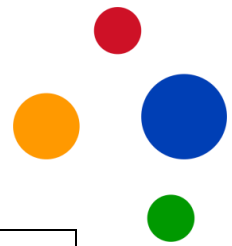
Students again start with connecting points and draw perpendicular bisectors for finding regions. Due to not very precise work, the triples of bisectors that should meet at one point, do not meet. The question arises what you do in the resulting area/triangle.



Students seem to have drawn some sort of perpendicular bisectors but did not construct them through precisely (e.g., the bisector between 2 and 5 is too close to point 2).



This seems to be a perfect solution. The strategy is not visible/evident.

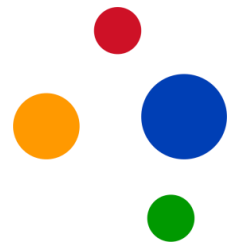


	<p><i>A clear solution with traces of the strategy used.</i></p>
	<p><i>Students draw piecewise straight lines by the rule of thumb. They acknowledged that they could not decide on some areas (see their question mark).</i></p>

From the observations we recognize four levels of achievement:

1. Drawing (curved) lines by the rule of thumb.
2. Using some mathematical argument (circles or midpoints) that does not lead to a strategy for the whole area.
3. Using perpendicular bisectors without precise constructions or not all bisectors, with some imprecision.
4. Realizing a construction with all relevant perpendicular bisectors and correctly determining all areas belonging to the wells.

Based on the present strategies in the classroom, the teacher needs to decide what can be validated and institutionalized for all students. It is important that the institutionalizing is based upon students' work and the knowledge shared by most of the students. Anyway, the teacher needs to challenge the students to identify ways by which less efficient strategies can be improved. The teacher asks students with different strategies/results to present their work and poses the question for the whole class: what are similarities and what are differences? How to improve and be sure? Can we describe a general strategy that everybody agrees upon?



Possible mathematical ideas to institutionalize:

- A conflict line between two points is the perpendicular bisector between these points.
- The perpendicular bisectors of three points meet in one point (unless these points are collinear).

Evaluation tools

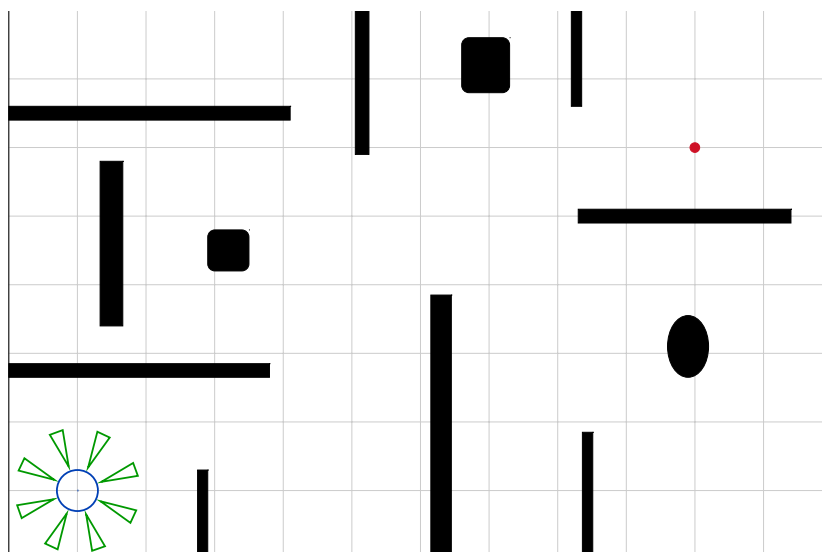
At the end of the lesson or soon after, the following tasks can be used for a quick test of the knowledge students developed during the scenario:

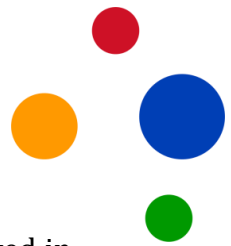
1. Give a situation with three points and conflict lines and a new fourth point and ask students to reconstruct the partitioning of the plane.
2. Give a situation with three points and conflict lines and ask them to create a situation with a fourth point in such a way that the result has not one or two isolated cells. When one of these is not possible, ask for an argument.
3. Let them describe and illustrate how to construct a partitioning of a plane with given points.
4. Ask them to explore Voronoi diagrams with three points in the applet and let them explain why they (almost) always see “three land points”.

Suggestion for further problems regarding conflict lines

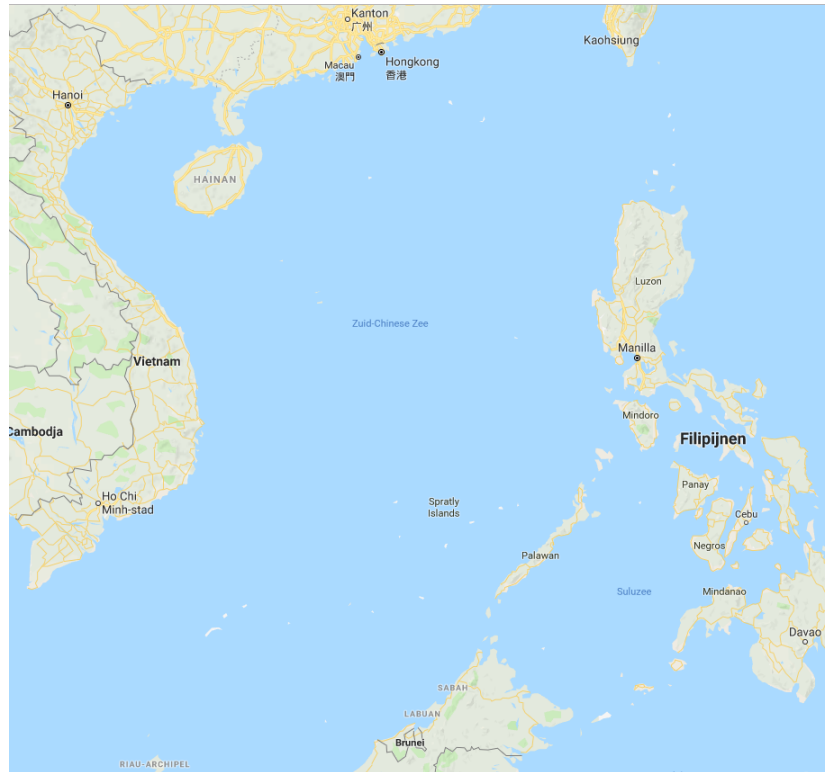
Several investigations in which (equal) distances to objects play a role are a territorial divide on sea and robot navigation, e.g. finding an optimal trajectory for a robot to move safely around or navigation on sea avoiding radar or guns.

1. A robot has to navigate through an apartment. What is the optimal path? How can the robot determine an optimal path through the apartment?

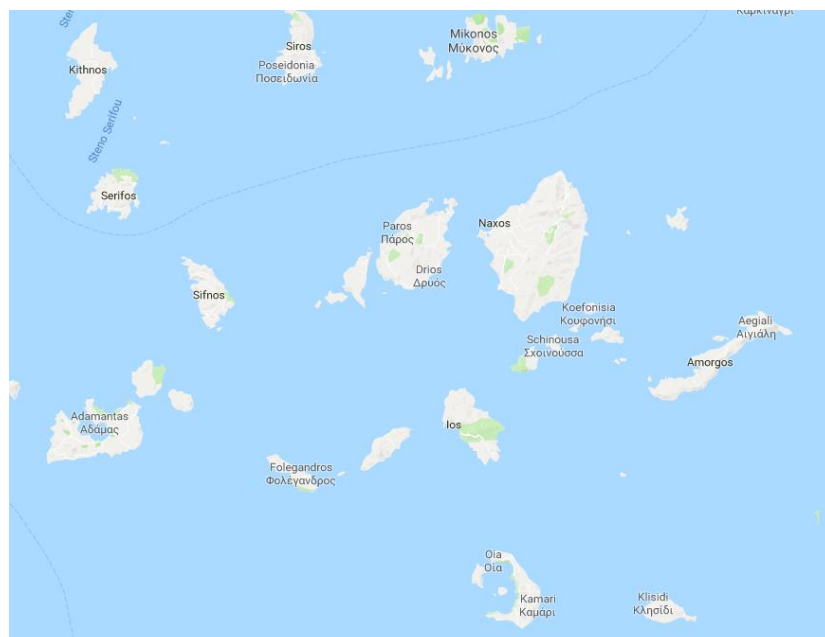


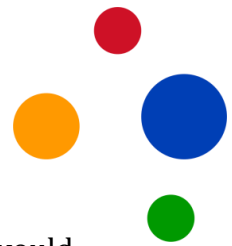


2. In the Chinese sea you can find quite a few islands. These islands are involved in territorial conflicts: how to draw borders on the sea to show what part of the sea belongs to what country? (also relevant because of resources that can be delved)



3. The Cyclades are an island group in the Aegean Sea. Its territory is divided into nine regional units of the South Aegean region by the biggest islands Andros, Kea, Kythnos, Milos, Mykonos, Naxos, Paros, Thira, Syros and Tinos. How would you divide the sea with the islands in the picture below?





4. Below you see the territorial division of the North Sea. Describe how you would underpin a possible division of the sea among the surrounding countries.



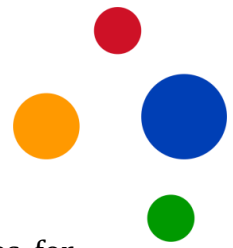
Rationale and RME perspectives on the scenario

Relevance and applicability

We consider the following three perspectives:

- *Real life*: this context connects to students' experiences with borders partitioning an area into regions and the notion of being closest. The concept of Voronoi diagrams is used in many different contexts close to the interest of students, e.g. football:





- *World of work:* Voronoi diagrams are an essential concept in many disciplines, for instance, biology (modelling cell structures), hydrology (calculating rainfall in areas), ecology (growth patterns of forests), chemistry (positions of nuclei in molecules), and computer science (spatial planning and robot control).
- *Further study:* Natural extension of the scenario is to consider the proof that point lies on the perpendicular bisector of a segment AB if and only if that point is equally distant from A and B (and hence lying on the Voronoi edge). The students may also investigate various problems of partitioning a geographical map or tracing out the optimal path for robot navigation. Another direction is to study cyclic quadrilaterals and other geometrical theorems and properties based on perpendicular bisectors (e.g. can you reconstruct a triangle when three intersecting perpendicular bisectors are given?). Yet another direction is to follow the scenario Conflict lines – parabola.

Inquiry skills

Students explore and investigate a real-life situation by trying and comparing different strategies. The solution relies on visualization and re-inventing the concept of the perpendicular bisector. To develop the winning strategy, the students have to make sense of the context and use geometrical language and notation.

The students are challenged to reflect critically upon the positions of the points and defend their reasoning about situations where three or more lines seem to meet. In the end, students need to present their result in a clear way and communicate about their strategies.

Potential for a sequence of lessons

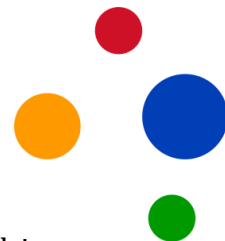
This scenario is a natural introduction for the study of conflict lines for various geometrical settings, or the study of more traditional topics in geometry related to perpendicular bisectors, triangles, cyclic quadrilaterals, etc.

- *Pre-knowledge:* To start the investigation students should have some familiarity with performing geometrical constructions with the compass and ruler for exploring the problem situation and creating a partition. To engage in the proof, students should be familiar with the triangle inequality for distance. Proofs and comparing different strategies require students to think abstractly.

Rationale for the scenario

- *Horizontal mathematizing:* The scenario invites students to model the contextual problem with mathematics. When solving the problem students need to realize which data is relevant in the situation (e.g. notice that the black rocks, dry grass and the actual distances are irrelevant) and then model the problem using geometrical notions: points, lines, distances, further and closer, perpendicular bisectors and a partitioning of the plane. This modeling activity brings the students in the world of mathematics where new issues, questions might emerge.
- *Vertical mathematizing:* The initial work of the students provides starting points for further mathematical exploration and generalization. Depending on the level and interest of the students, one of the emerging issues that require proof is that the points equally distant from two fixed points all lie on the perpendicular bisector and think about its construction.

Furthermore, the need is created for investigating patterns emerging with three points. Their perpendicular bisectors seem to meet at one point. With four points



other patterns become visible. From these issues the teacher can proceed into textbook topics that address perpendicular bisector theorem for triangles, and further geometrical investigations like the circumcenter and cyclic polygons. Another direction for follow-up lessons can be to use the notion of conflict line for exploring definitions and characteristics of parabolas (see the MERIA scenario Conflict lines – parabola).

Conclusion

This scenario illustrates how the problem in the desert context, that is rich and accessible for all students, can be used to elicit initial solving strategies and representations like concentric circles, midpoints and dividing lines with language like “being closest”. These elements can be used to build on the mathematics of perpendicular bisectors and partitioning a plane through processes of horizontal and vertical mathematization. By learning these notions from applications, students might easier see the potential of applying them in other situations.



MERIA Module “Job Advertisement”

Measures of central tendency

The teaching scenario

Target knowledge	Determine, distinguish between and make decisions on measures of central tendency (arithmetic mean, mode, median).
Broader goals	Analysing data. Drawing histograms and other graphical representations (plots), as well as calculating statistical measures on paper or using ICT. Understanding the issues and misconceptions that arise in statistics. Inquiry skills: making and evaluating decisions based on arguments, comparing different ways of reasoning, interpretation of data and formulation of conclusions. Interdisciplinary skills: students can connect statistical problems to everyday situations and situations in economy. They learn to appreciate the use of mathematical reasoning in decision making.
Prerequisite mathematical knowledge	Calculating arithmetic mean. Familiarity with the notion of average. Basic skills in the use of ICT: manipulating Excel spreadsheets or similar (e. g. Google Sheets or OpenOffice); knowing how to use basic commands to compute sums and averages; representing data graphically (histograms, scatter plots, box plots ...)
Grade	Age 15 - 18, grade 9 - 12 (whenever the arithmetic mean is introduced)
Time	45 minutes (could be extended to 90 minutes)
Required material	Computer, appropriate software (Excel, Google Sheets, OpenOffice, GeoGebra ...). Data set, referred to in the following as the ‘payroll’. The data set is appended to the scenario as an Job_advertisement_data.xlsx file.

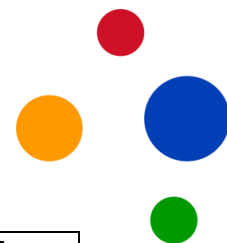
Problem:

Companies advertise for new employees. To give prospective applicants an idea of the income possibilities in the company (‘boast about the company’) the advertisement informs the applicants of the average monthly pay. In the material you have the payrolls of these three companies.

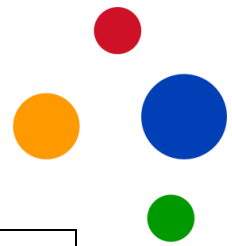


In which company would you seek employment?
Explain and give mathematical reasons for your decision.

Consider the following: Which salary divides the employees into two groups of the same size? Which salary would be the best representative of the payroll?



Phase	Teacher's actions incl. instructions	Students' actions and reactions
Devolution (didactical) 5 min	The teacher presents the problem to the students and gives them a link to the <i>Excel</i> sheet with data (3 payrolls). (S)he suggests using technology (data analysing and graphing tools) to help them with reaching a decision. The teacher organizes students in groups of two or three.	Students listen and ask questions.
Action (adidactical) 20 minutes	The teacher circulates and observes, helping just in case of some technical difficulties (not with the use of the program). (S)he notes the different strategies the students choose.	Students discuss in their groups what technology they will use, what "mathematics" they will use, and how to organize work.
Formulation (adidactical) 5 minutes	The teacher asks the students to organize their process and formulate decisions.	Students organize and summarize their work.
Validation (didactical / adidactical) 10 minutes	The teacher chooses some students to shortly present their solutions – decisions. Groups with different strategies should be chosen.	Students give short explanations of what they were doing. Other students listen and discuss.
Institutionalis ation (didactical) 5 minutes	Summarize students' work and generalize: How to choose the number that best represents the data set. The teacher defines the measures of the central tendency - the arithmetic mean, the median and the mode, and how they are determined. (S)he summarizes the influence of data on the arithmetic mean, median (and mode), advantages and disadvantages of each measure. Be clear that this situation does not have one answer, but the result should be the different information that each of the measures provides.	Students listen and ask questions.

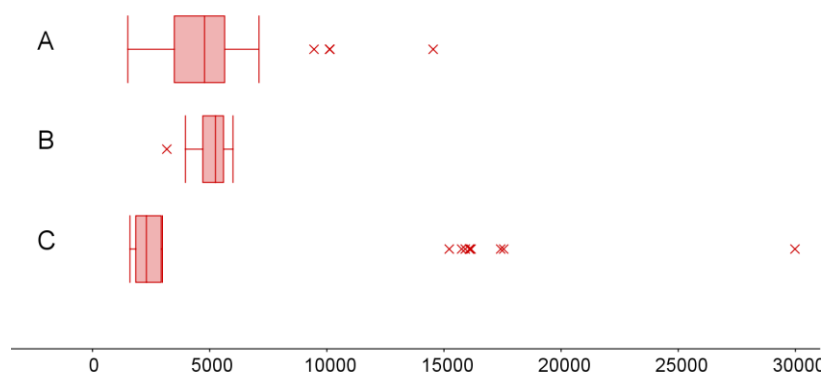


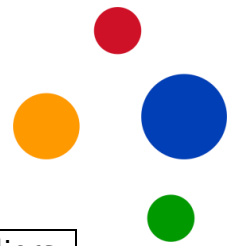
Possible ways for students to realize target knowledge

- Arithmetic mean and median:
 - Some students might immediately know what to do so they start graphically representing data using familiar technology and using data analysis tools for calculating the arithmetic mean and median for each payroll. They will compare the lists and notice how outliers (big data) affect the mean, and consequently reach the decision which company to choose.

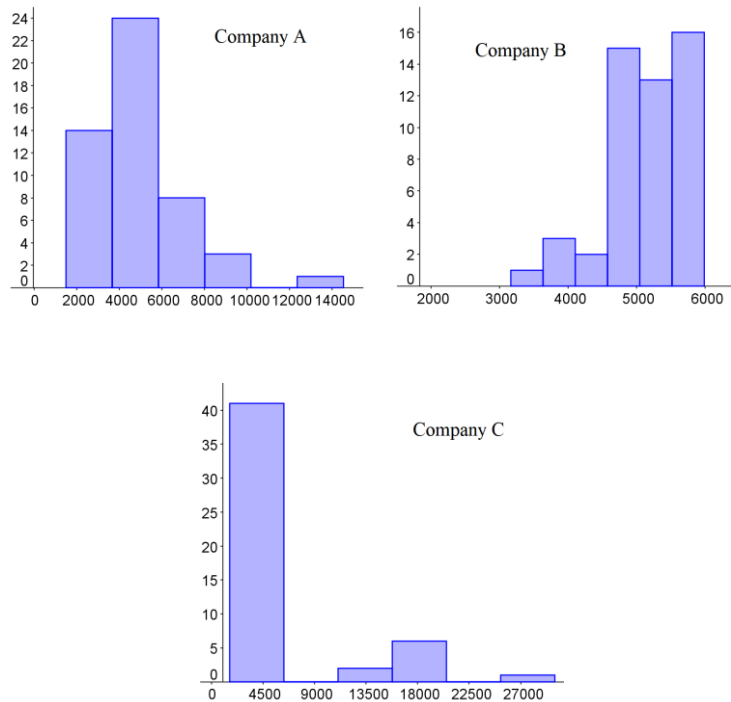
	<i>Company A</i>	<i>Company B</i>	<i>Company C</i>
Mean	4939.98	5138.04	4992.6
Median	4774.5	5241	2293.5
Range	13038	2826	28394
Minimum	1500	3165	1593
Maximum	14538	5991	29987

- Some students will observe tables, sort the data and discover how to find the middle data (median) on their own. They will notice that in tables with sorted data, especially in payroll C, there are some outliers (in relation to the rest of the data set) and investigate what happens with arithmetic mean and median with and without them. Consequently, they will learn the advantages and disadvantages of each measure.
- Some students will only graphically represent data and make conclusions from graphs. These students might use box and whisker plots where they can read out all the information they need (arithmetic mean and median) and make a decision. In addition, the outliers are easy to spot in box plots and they will conclude how they affect the arithmetic mean and the median.





- Some students will draw histograms and notice the outliers. From the histograms, they will notice the influence of the outliers on the averages.

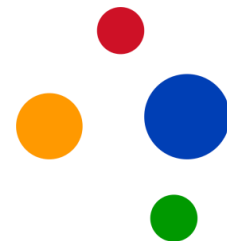


- As for the mode, in order for students to be able to determine it, they can be encouraged by the teacher to round the data or group them into classes. Then they could represent the new data graphically in some form, which includes frequency classes (e.g. histogram). After that, they will compute mode for each payroll (or read the value of it from e.g. histogram) and enhance their previous decision regardless of which method they used to determine the arithmetic mean and the median.

Explanation of materials

Students will be given a data set (“payrolls”) with three lists of monthly salaries for 50 employees and calculated average salary. The data is not sorted. Students are expected to come to the idea to sort the data and draw some diagrams.

To make a task more attractive to students, companies could be named with attractive names, or the task can be presented in the form of an advertisement published in the newspapers by three companies.



Variations based on didactic variables

The data set could be given in the digital form, or both in digital and in paper form. The paper form can be used for introducing the problem and first brainstorming, but technology is obligatory to be used in this scenario.

Variations could be done in the *organization of class work*: pairs or groups of three. Larger groups are not recommended because of working with technology.

A small variation in *time* is possible in the action and formulation phase.

In order to keep the target knowledge, it is recommended not to change the data set.

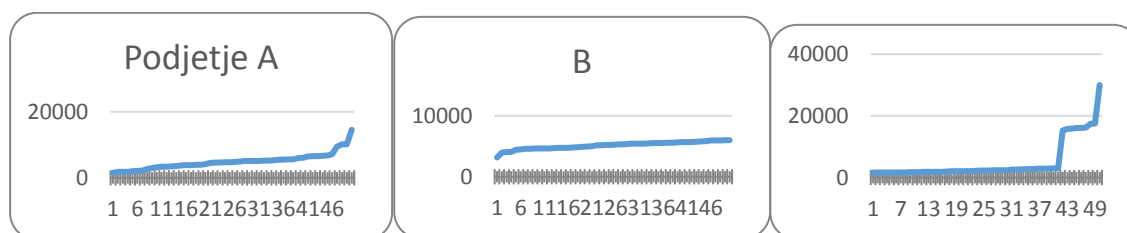
In case students have difficulties using Excel or other technology to sort and manipulate the data, the data sets can be sorted in advance. Another option is to precede this activity with a short introduction to Excel.

If students are already familiar with measures of central tendency, possible extensions in target knowledge could be:

- Introducing measures of spread (variance)
- Quantiles included in measures of spread
- Graphical representations (boxplot)

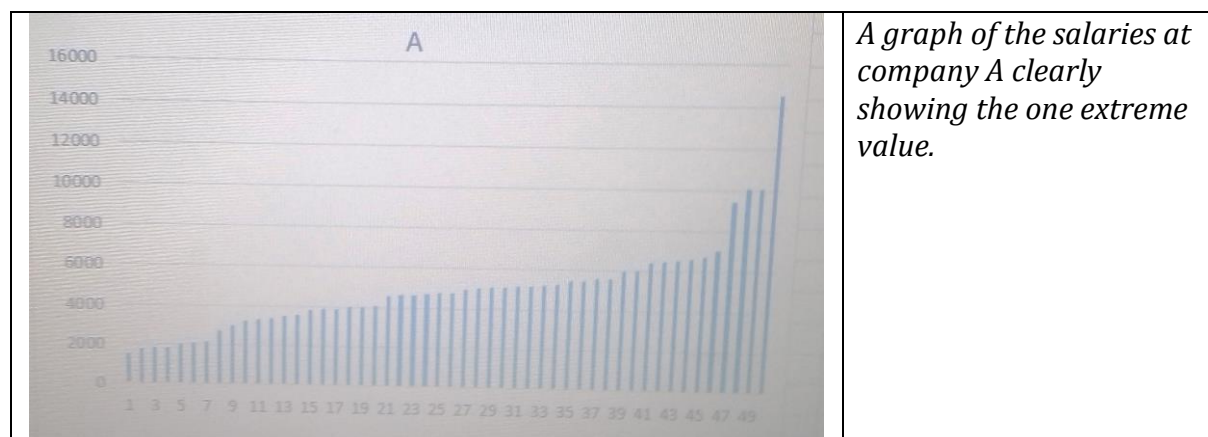
Observations from practice

Students sorted the data and presented it graphically in the following way:

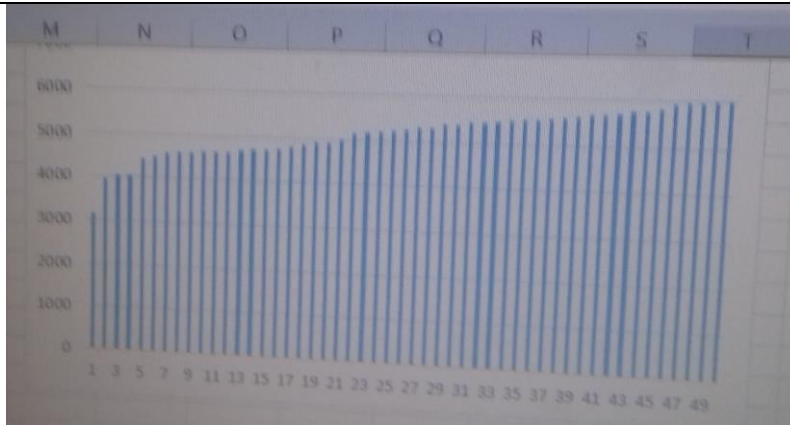
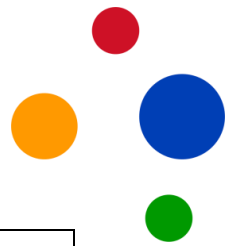


The x-axis represents employees and y-axis salary.

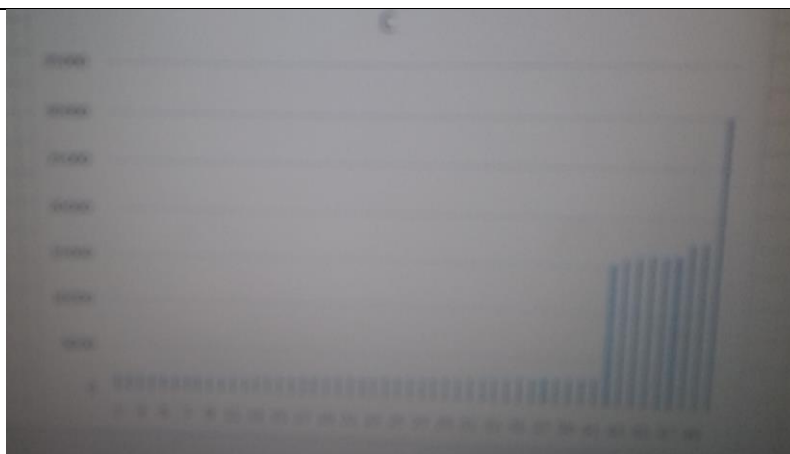
Other graphical representations of the data put the employees on the horizontal axis and the salaries on the vertical axis. This is what Excel suggests as a graph, and it also gives information about the differences between the companies:



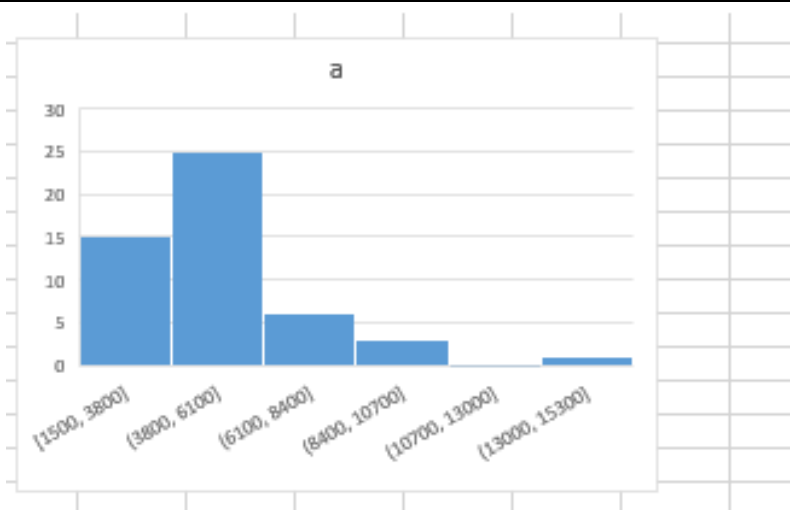
A graph of the salaries at company A clearly showing the one extreme value.



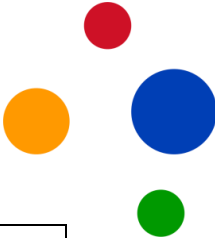
A similar graph of company B showing a more constant spread with a few lower salaries. One of the students favoured this company for a first job since immediately you earn quite some money.

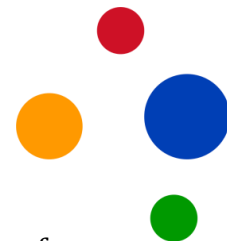


The graph of company C shows a large number of low salaries, a few higher ones and one extreme value. An ambitious student said he would join this company.



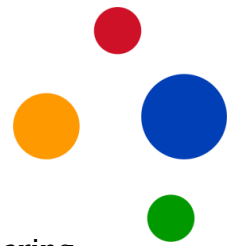
The student had some prior knowledge in analysing data. She grouped the salaries of each company and used a histogram to represent it. In addition, she used "box and whisker" plot to represent data and make a decision.



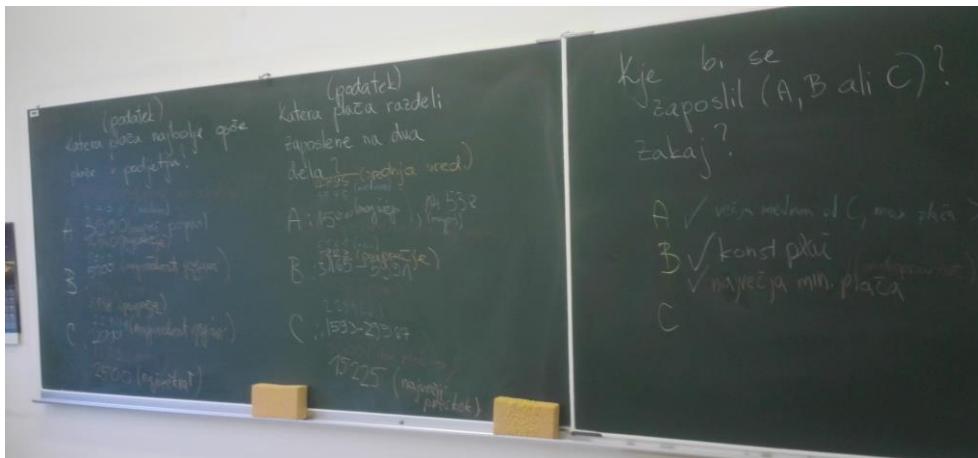


When students were asked to formulate the results of their calculations of measures for central tendencies, they get different numbers:

<p>odločili smo se za B, ker je najmanjša plača 3165.</p> <p>Najbližja povprečni plači:</p> <p>- A: 5064 → A: najbolj opredeli</p> <p>- B: 5160 B:</p> <p>- C: 2979 C:</p>	<p>The students were looking for the salary in the dataset that is closest to the given mean value as the best choice for representing the dataset. Their best choice is B because the lowest salary in that company is the highest.</p>																																																																																																																																																																																																																																																																																																																																																																
<p>Povprečna plača:</p> <p>PODJETJE A: 4940 €</p> <p>PODJETJE B: 5138 €</p> <p>PODJETJE C: 4992 €</p> <p>Kje bi se zaposlili?</p> <p>V podjetju B</p> <p>Zaraj?</p> <p>ker je manj odstopanja od povprečne plače.</p>	<p>Students recall the given mean values and their choice is B because the salaries in that company have the smallest differences from the mean value.</p>																																																																																																																																																																																																																																																																																																																																																																
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students highlighted salaries that divide the list into two equal parts and found the median by using Excel. They choose company B because it has the highest average pay as well as the biggest median.</p>
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The teacher used these different calculations for comparison and discussion by sharing them on the blackboard:



Evaluation tools

1. Determine the mean, median and mode of the following set of the data:

a) 4, 4, 4, 5, 6, 6, 20

b) 4, 4, 4, 5, 6, 19

Answer:

a) Mean = 7, Median = 5, Mode = 4

b) Mean = 7, Median = 4.5, Mode = 4

2. In a company almost all workers have the same salary except for the manager who earns ten times as much. What is a good measure for central tendency? Explain your answer.

Answer: Median because outlier (big salary) has a big influence on mean value.

3. Following data are test results (out of 100) for 45 students.

59	32	81	70	71	72	83	92	95
61	69	59	91	84	73	74	66	77
70	67	65	58	59	78	93	95	50
62	67	92	65	54	90	92	79	62
75	83	98	71	83	67	59	46	64

a) Find the mean, median and mode of the measurements.

b) Do you think the median represents correctly this set of data? Explain.

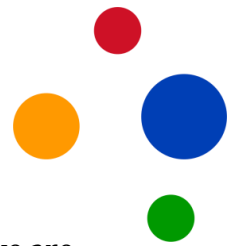
c) Your result is 58. What measure you will use for comparison when you will report results to your parents?

Answer:

a) Mode: 59, Median: 71, Mean: 72.3

b) A better measure is median if we state that 32 points are outliers.

c) The mode is the best measure if a student wants to present that his result is not far away from the central value.



4. You have to choose a location for summer holidays, and the only data you have are measures of central tendency for daily temperature in 90 days of summer, measured in C degrees at noon:

	A	B	C
Mean	32.5	32	33
Median	26	32	26
Mode	20	31	26

What do these numbers tell about the climate in each location?

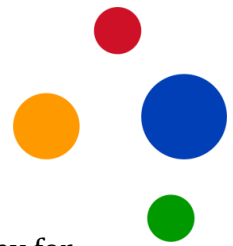
Suggestion for further problems

A next problem could repeat a similar activity in another context: Given lists of students' grades. Which measure of central tendency would better summarize their grades? Another activity could move towards the use of representations like graphs, boxplots and histograms and include the notion of spread, for instance by providing graphs with salaries instead of tables with raw data and setting the same task as in this scenario. They could also be asked to find situations in which either mean, mode or median is the best measure to represent the central tendency.

Examples:

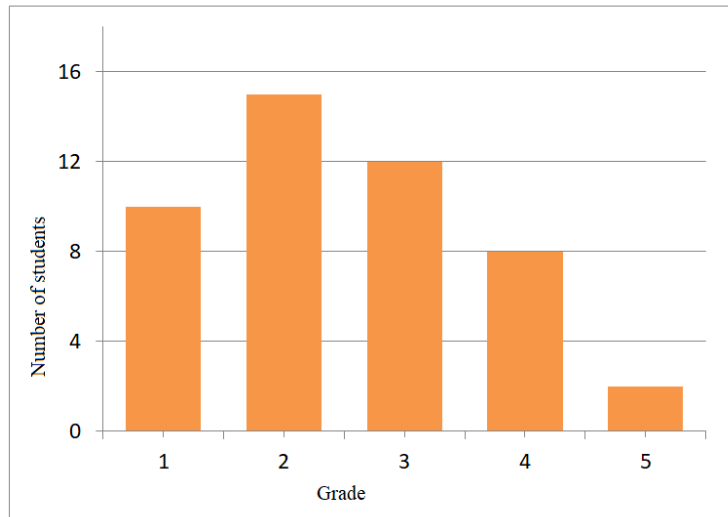
1. You have to make a decision on how to go to school: by bus, by train or by bike. You found some data about your city traffic company. Some data about travel times in minutes, rounded to whole minutes, from your home to school are given based upon experiences during the day. You are lucky, your home and your school both are close to the bus and train station. Data are based on the measurements that have been taken during 12 months. In addition to these data, you can also consider other variables. Make a decision about your transport and provide arguments.

		By bus	By train	By bike
Monday	mean	20	12	19
	median	14	13	19
	mode	15	12	19
Tuesday	mean	19	18	19
	median	13	12	18
	mode	14	13	18
Wednesday	mean	18	12	18
	median	12	13	17
	mode	14	12	18
Thursday	mean	19	14	18
	median	12	12	18
	mode	15	13	18
Friday	mean	24	19	20
	median	20	14	19
	mode	19	15	20



Students should discuss situations from their own point of view. They can say for example that they would go by bus except on Friday, or that decision depends on the season. In addition, they can expect that buses run more often than trains, furthermore for the bike you have no time for waiting. They can also take prices into account.

2. The following graph gives a number of students that achieve a certain grade.



- Which of the measures of the central tendency can be found easily from the graph only?
- Determine all measures of the central tendency and justify your procedure.
- Describe and comment test results using measures of the central tendency.

Rationale and RME perspectives on the scenario

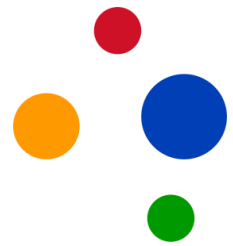
In this scenario measures for central tendency emerging from exploring and structuring experimental data on payrolls.

Learning from and in applications enable students to better see how to apply their statistical knowledge and skills. This scenario contributes to the ability to reason with statistical results and to interpret them critically.

Follow up lessons could further elaborate on measures of central tendency in other contexts and include measures for the spread and graphical representations like boxplots and histograms.

Relevance and applicability

Learning how and when to apply is especially important for statistics since it is used in many different disciplines (e.g. experimental physics, social sciences, chemistry, psychology, medicine ...). In addition, in daily life students are also often confronted with statistical results (e.g. in newspapers, grades at school).



Inquiry skills

The inquiry is included in all the phases. Students should be accustomed to investigating and more often put into situations where they will work in this way. Thus, while developing mathematical competence they also develop inquiry skills. During the scenario implementation students will experiment systematically, organize data, make decisions, collaborate and communicate. Inquiry skills should be included in the institutionalization, especially the skill of organizing, structuring and summarizing data.

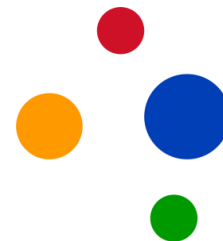
Potential for a series of lessons

The scenario can be part of a larger set of lessons on statistics and data visualization.

- *Preliminary knowledge:* Students are expected to be able to use Excel for some elementary data manipulation like sorting, representing and performing calculations. Furthermore, we start from the arithmetic mean as preliminary knowledge of the students.
- *Introduction:* A problem can be introduced by introducing a person, just finished university and looking for a job. She reads a newspaper with job advertisements for three different companies that sound interesting. She searches for payroll information at those companies and compares the earning options at each of the companies. How to make a decision?

Rationale for the scenario

- *Horizontal mathematizing:* The context supports students in using their initial language to describe characteristics of the data for mathematizing the problem. Words being used are: extremes, order, range, many almost the same, many differences. They also feel the need to graph the data or organize it in intervals, since the large set is difficult to oversee or analyse. This language and representations help them to enter the world of statistics and connect it to a realistic situation.
- *Vertical mathematizing:* The expected variety in reasoning by the students can be used in the validation and institutionalization phases to develop the formal statistical measures for central tendency, the ways to calculate them and to use them. Further study could be oriented on relations between graphs of data and measures of spread in connection with these measures of central tendency, and also how to effectively use technology in statistics. On a meta level, it is also important to involve students in how statistics is being used to summarize data and to make decisions and predictions about the world around them.



MERIA Module “Slide”

Introduction to derivative

The teaching scenario

Target knowledge	Conceptual understanding of the slope of a curve as the slope of the tangent line.
Broader goals	Mathematical modelling of the slide using graphs of functions. Calculating the slope (the derivative of a function) by hand or using ICT. A meaningful introduction to calculus. Inquiry skills: experimenting with different graphs of functions on paper and using ICT, iterating a process to improve the solution, comparing different strategies, justifying the characteristics of the obtained solution. Interdisciplinary skills: students can connect their experience of the smoothness of physical objects to mathematical terms of the tangent to a curve and the derivative of a function. The mathematical models may be used to produce 3D objects by printing on a 3D printer (ICT skills) or by production with other materials (crafts).
Prerequisite mathematical knowledge	Graphs and equations of a line and some nonlinear curves (circle, parabola or graph of the exponential function).
Grade	Age 16 - 18, Grade 10 - 12 (whenever derivatives are introduced)
Time	60 - 90 minutes, two lessons
Required material	Paper, pencil, ICT - a tool for graphing functions, such as GeoGebra (the use of ICT is strictly speaking not needed but may greatly enhance the students' experience).

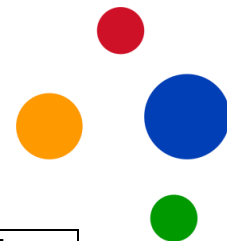
Problem:

Look at the pictures of a ski jump in-run and a children's slide. Both have a curved part at the bottom and/or the top and a straight part in the middle. Use mathematics to design such a shape. Focus on just one of the curved parts and the straight part in the middle. Remember, it is not nice to have a bumpy ride. Introduce a coordinate system and find equations for *one* curved part and the line.

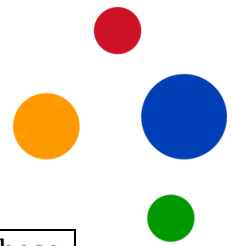


The [Holmenkollen ski jump](#) in Oslo, Norway. Photo taken by [Mathias Stang](#) and a children's slide.

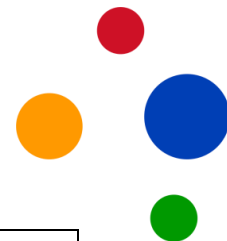
Note: For a longer lesson, with more modelling activity, omit this last sentence from the task description (see the module for extra lesson phases).



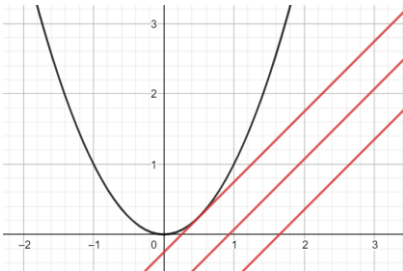
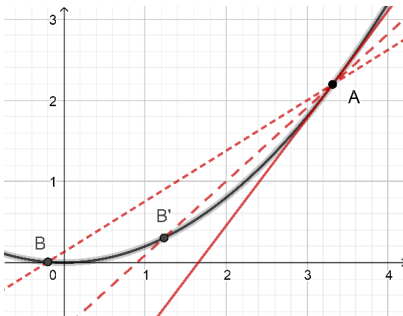
Phase	Teacher's actions incl. instructions	Students' actions and reactions
<p>Devolution (didactical)</p> <p>5 min</p>	<p>The teacher introduces the problem.</p> <p>The teacher points out that the students should design a smooth ride of a slide.</p> <p>The teacher makes sure that students focus on just one of the curved parts and the straight (linear) part in the middle.</p>	<p>Students sit down in groups of two or three.</p> <p>Students get excited!</p>
<p>Action (adidactical)</p> <p>20 min</p>	<p>The teacher registers the students' ideas, strategies, and findings.</p> <p>If students do not realize that the two parts should connect smoothly, the teacher should address this matter.</p> <p>If there are absolutely no ideas for the choice of the curved part after 10 minutes, the teacher reminds the students what the graphs of $y = x^2$ and/or $y = \cos x$ look like (not the circle), during a brief whole class (didactical) interruption.</p> <p>If students came up with the circle solution, one of the follow-up problems is: "What if you change the angle or what if you change the point in which the line and the circle meet? How does the equation for the line change?" After that, the teacher asks the group of students to focus on the case where the curve is not a circle.</p>	<p>The student makes a sketch and introduces a coordinate system.</p> <p>Students' approaches can usually be described by one of the following categories:</p> <ol style="list-style-type: none"> 1. Bounding line approach: they choose a free line and then move (translate and rotate) it until there seems to be just one intersection point in the area of focus. 2. Secant lines approach: they choose one point on the curve: the intended point of tangency; then another point on the curve, draw the line between the two points and move the second point closer to the first to obtain a smoother fit. 3. Linear approximation approach: Students choose one point on the curve, draw a line and then try to adjust the slope so that it fits best against the curve. <p>Some may use a circle as a curve and the fact that a tangent is perpendicular to a radius. We call this the <i>circle solution</i>.</p>

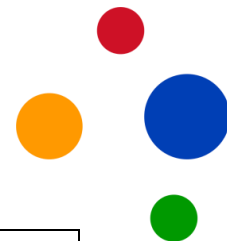


		See below for details on these (categories of) approaches in the section <i>Possible ways for students to realize target knowledge</i> .
Formulation (adidactical) 15 min	The teacher asks the students to formulate their results. While they work on this, the teacher chooses groups with different approaches who will present their findings.	Students formulate their results within their group. For some groups, a student presents their findings.
Validation (didactical) 10 min	The teacher asks: “When do we know that the solution is good?” and “Is there a best solution?” If students have used visual validation only, the teachers could suggest algebraic or numerical approaches for validation.	They explain why some solution is good and whether one might be better than another. <ul style="list-style-type: none"> • Visual validation: Some will rely on their visual evaluation of the design; if it looks good, then it is good. They may also zoom in on the curve. • Algebraic validation: The students may compute intersection point(s) algebraically and perhaps see it is locally unique. • Numerical validation: Students can compute $\frac{\Delta y}{\Delta x}$ for two points on the curve and see if it is approximately the slope of their line. <p>If the students have worked on a circle solution and computed the tangent line, they should be certain that they have a tangent and explain why (geometric and/or algebraic proof).</p>
Institutionalisation (didactical) 10 min	The teacher discusses the notion of the tangent line in the way that matches what the students came up with. The teacher can highlight one <i>or more</i> of the following viewpoints on the slope of the curve in a point:	Some may say something about the slope. Some may use the word “tangent” or the button in GeoGebra. Students listen and become interested in computing the best solution to the problem of arbitrary shapes and curves.

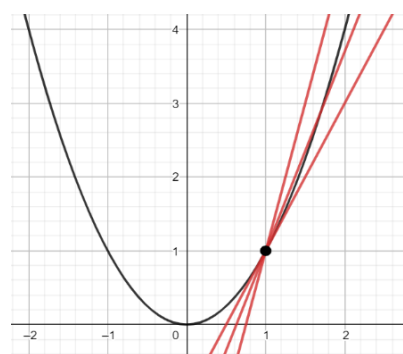


	<p>a) best local approximation follows visual validation</p> <p>b) locally unique bounding line - one intersection point follows algebraic validation</p> <p>c) classical definition using secant line and limits of difference quotients follows numerical validation</p> <p>If a circle solution comes up, a tangent to the circle and a tangent to other curves are discussed. The teacher recalls that the best solution for the circle is the tangent and that the students have actually approximated the tangent for the other curves.</p>	
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<p>Possible ways for students to realize target knowledge</p>	<p>There are different options as to what the students do, for example:</p> <ol style="list-style-type: none"> <p>Bounding line approach: Students choose for example $y = x^2$. Algebraic validation: from here consider the family of lines $y = x + b$. The bounding line is found by y-elimination: $x^2 = x + b$. This equation has a unique solution if the discriminant equals zero: $1 + 4b = 0$. So $b = -\frac{1}{4}$ gives a smooth slide.</p> <p>Secant line approach: Students fix one point on the curve, the intended point of tangency. Then they choose another point on the curve, draw the line between those two points and move the second point closer to the first to obtain a smoother fit. The closer you chose points the better approximation will be. This approach works best with ICT.</p> 	 
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3. Linear approximation approach:
For example, students work with $y = x^2$ and the point (1,1) where the curve ends and the line $y = ax + b$ begins. Then they might guess $a > 1$ and try various values (of which $a = 2$ is correct). Trying means by drawing or graphing. Describing the line by $y = ax + b$, they derive that $a + b = 1$.

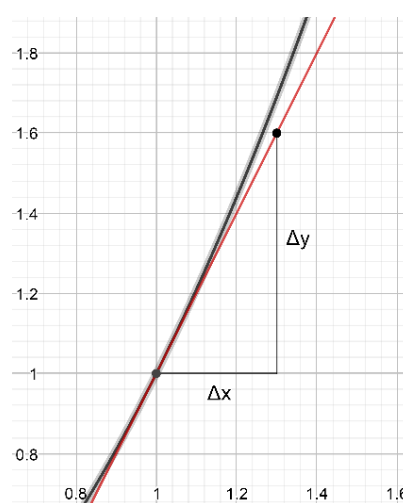


So with each slope a , they can compute b .

Some may determine an approximation of a using two points on the drawn line using $\frac{\Delta y}{\Delta x}$.

Numerical example: students may find $a = \frac{\Delta y}{\Delta x} = \frac{0.6}{0.3} = 2$. From $a + b = 1$ follows $b = -1$.

Validation is probably done visually, but it can be done numerically, possibly suggested by the teacher, because the method is similar. Now choose two points on the parabola and compute $\frac{\Delta y}{\Delta x}$; for



example (1,1) and (1.1, 1.21). Then $\frac{\Delta y}{\Delta x} = \frac{0.21}{0.1} = 2.1$. Pretty close!

Students can also validate by computing the intersection point of the line and the parabola (algebraic validation). If students are familiar with quadratic equations and the discriminant they can proceed by solving the system of equations:

$$y = x^2, y = ax + 1 - a,$$

and get $x^2 - ax + a - 1 = 0$.

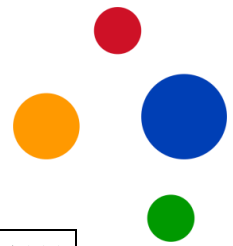
The equation will have exactly one solution if the discriminant is equal to 0:

$$a^2 - 4(a - 1) = 0 \Rightarrow a = 2.$$

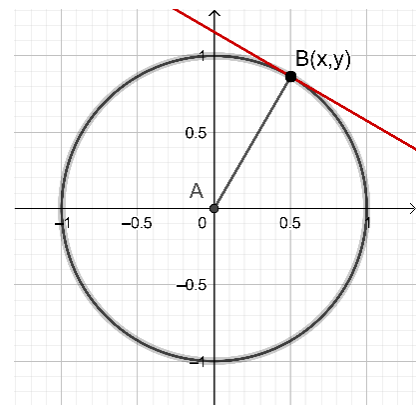
4. Circle solution: Students choose a circle.

If the students have chosen a circle, they might choose for example $x^2 + y^2 = 1$ and the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, that corresponds to the angle $\frac{\pi}{4}$.

If they know that the radius of the circle is perpendicular to the tangent, they could determine $a = -1$. Then they could determine the equation of the tangent line.



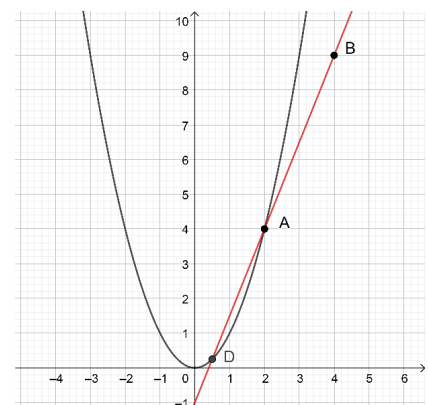
If the teacher asks them to choose a different point (x, y) , they could determine the slope a of the tangent line from the slope of the radial line through (x, y) , which is $\frac{y}{x}$. So $a = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$ (in general), but probably students will do this for one concrete point. This consideration may become a bit simpler if the students know and use vectors.

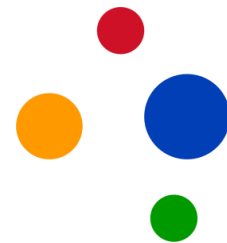


5. With ICT (GeoGebra or similar)

If the students use GeoGebra (or some other ICT), they would probably use similar steps and reasoning as without. The difference is that the ICT computes the equation of the line faster and draws an accurate representation of the chosen curve(s). With ICT the students can try more options in less time and can thus notice things, they would not with pen and paper. For example:

- Some may find and use a button for the tangent line.
- They might draw the curve and an arbitrary “good” line through a point on the curve and some other point. Then they zoom in and check whether it looks okay. They could move the second point to get a better fit. They could choose one possibility that they find is the best one and read the equation of the line by a “measuring tool”.
- Some students may begin by zooming in on a point until the graph of the curve looks straight. Then they could choose two points on it to compute an equation for it (or at least to draw a more or less tangent line).
- They might try to see if their line has intersections with the curve (in this case it matters if they have drawn a half line or a line). Some might even let GeoGebra show the intersection points of the line and the curve and they could notice that when they change the slope of the line with a fixed point on the curve, they also change the (other) intersection with the curve (as pointed out in the Institutionalisation step). Because they immediately see the result, they could arrive at the hypothesis that the best solution is when points A and D coincide.





Explanation of materials

The idea of designing a familiar shape should excite the students in the devolution phase and introduces a real-life context where smoothness in an intuitive sense is important. If some students are not comfortable with the ski jump or children's slide shapes, the teacher could tell them the same principles apply when constructing train rails, rollercoasters, etc. (focus on connecting one curved part with a linear part). This also conveys to the students that the problem is real.

Apart from possibly ICT, there are no particular materials needed for this scenario.

Variations based on didactic variables

Some changes that can be made in the scenario (without changing the objectives) are:

The milieu: the pictures can be chosen differently, but should contain a curved and a straight part. Preferably, the object to design consists of one curved and one linear part. It should be as familiar to the students as possible.

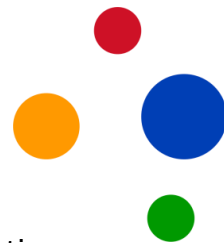
In some cases, the first action may find students stuck after a few minutes, or otherwise not involve productive mathematical work. The teacher can then interrupt the work as follows: The teacher can add an interlude in the action phase to discuss students' first findings and challenges, focusing on what a smooth fit might mean.

If the teacher chose to omit the phrase "Introduce a coordinate system and find equations for *one* curved part and the line." from the devolution (task description), then she could add an interlude to discuss the necessity of this for the mathematization of the problem. At the end of this interlude, the students should understand that they need to work with coordinate systems and concrete functions/equations for the curved part and the line. Before continuing the teacher checks whether the students have an idea of what a good and a bad line fit would be geometrically (the solution requires not only continuous but also a smooth curve). After this interlude, the students continue with the action phase as described in the scenario.

The students could use geometric software: e.g. GeoGebra, Geometer's Sketchpad, Desmos, Wolfram Alpha (or Mathematica), Maple, MATLAB, Octave, etc. Alternatively, they could use a graphical calculator or mobile phone with geometric software. We recommend giving students the option, but the choice should be left to the students.

The length of the phases can be adapted to the students' work, but should not change too much.

In the *action phase*, the students should be left to choose their own equation for the curved part. Only if some groups (or the whole class) have no idea on how to proceed after 10 minutes, the teacher can remind those groups (or the whole class) of some options: e.g. $\cos x$, $\frac{1}{x}$ or x^2 (but not the circle). After the students choose a function for the curved part, the action phase continues.



If only a few students have a circle solution, the teacher can pose the follow-up questions as explained in the scenario in that group alone, to avoid disrupting the chain of thought of the other students. If there were no students that came up with the circle solution, the teacher could mention it at the validation or institutionalization phase, but not before.

Observations from practice

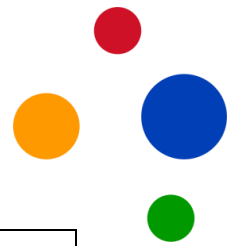
Some general observations:


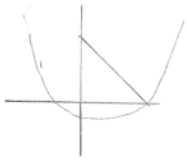

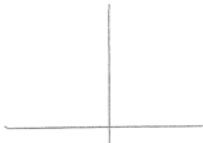
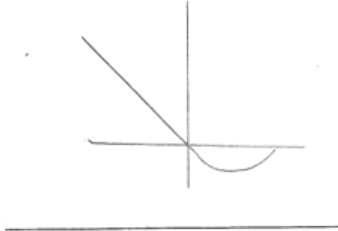
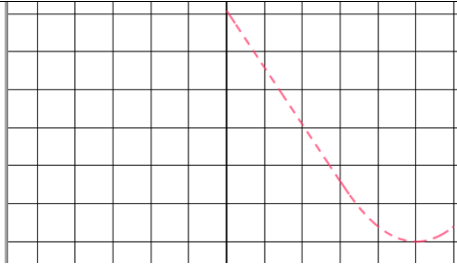
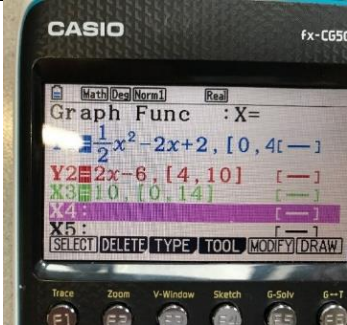
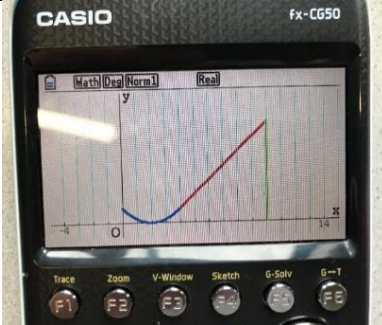
- In general, teachers and students liked this lesson.
- Sometimes students were focused too much on the shape of the curved parts, so the teacher had to remind them that the design also had to include a line segment, and to join the curved and linear parts.
- Some students worried about the practicalities of the slide other than the smoothness, so the teacher should really check whether the students have an idea of what a good and a bad line fit would be in a geometrical sense.
- Some teachers and students used ICT, e.g. GeoGebra, Geometer's Sketchpad, Desmos graph mobile application, Graphic calculator. The groups using ICT explored more ideas. Some students begin on paper and check with ICT. Some work the other way round.
- In the tests, some groups of students had already learned derivatives and most of those realized that they must use the tangent for the solution.
- Some students had a discussion on what to vary: parabola, a slope of the line, or intercept of the line.
- Some didn't realize they could introduce a parameter somewhere: e. g. a or b in $y = ax + b$ or a, b or c in $y = ax^2 + bx + c$, or even Δx as a parameter.

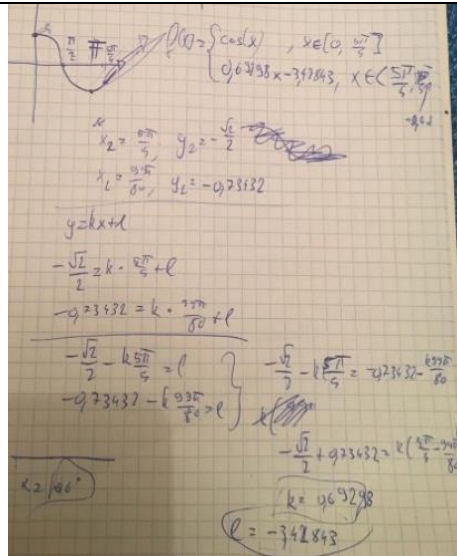
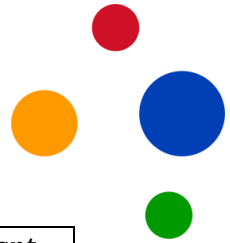
The students' design approaches are related to different aspects of what they think a tangent line is.

1. Bounding line approach: they choose a free line and then move (translate and rotate) it until there seems to be just one intersection point in the area of focus.
2. Secant lines approach: they choose one point on the curve: the intended point of tangency; then another point on the curve, draw the line between the two points and move the second point closer to the first to obtain a smoother fit.
3. Linear approximation approach: Students choose one point on the curve, draw a line and then try to adjust the slope so that it visually appears to "fit" the curve.

Some used a circle as a curve and the fact that a tangent is perpendicular to a radius. We call this the *circle solution*.



<p>Invoerfunctie(s) $\frac{1}{4}x^2$ Resultaat: $5-2x+10$ $[0,5]$</p> 	<p>Invoerfunctie(s) $\frac{1}{4}x^2 - 1x - 1$ Resultaat: $-2x+10, [0,5]$</p> 	<p>Students vary the parameters of the parabola and the line. In the end, they find a good solution. They seem to focus on intersection points and evaluate visually.</p>
<p>Invoerfunctie(s) $\frac{1}{6}x^2 - 1x$ Resultaat: $-2x, [0,5,0]$</p> 	<p>Invoerfunctie(s) Resultaat:</p> 	
<p>Invoerfunctie(s) Resultaat:</p>  <p>$\frac{1}{3}x^2 - 2x, [0,6]$ $-2x, [-5,6]$</p>		
<ul style="list-style-type: none"> ● $h(x) = 0.2(x-10)^2 + 4, (6.05 \leq x \leq 12)$ ● $p(x) = -1.5x + 16.17, (0 \leq x \leq 6.05)$ Line f: $1.5x + y = 16.17$ Point A = (6.05, 7.1) 		<p>This group just varies the b in $y = ax + b$ and finds an approximate solution.</p>
		<p>A solution is shown on the graphical calculator.</p>



This group used the secant lines approach (2): forming a line through two near points on the curved part for the approximation of the linear part. They used numerical methods to find an equation for the line: they write that they choose a point where the cosine ends ($x = 5\pi/4$) and then one point a little bit before that one ($x = \frac{99}{100} \cdot 5\pi/4$).

[Ikkokovi dvije bliske točke na grafu kosinusa računski]

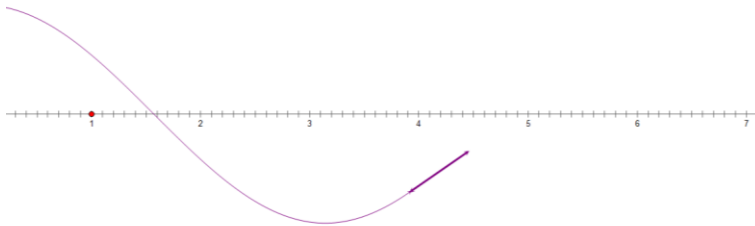
struct Transform Measure Number Graph Window Help

Skakaonica

$$g(x) = \cos(x)$$

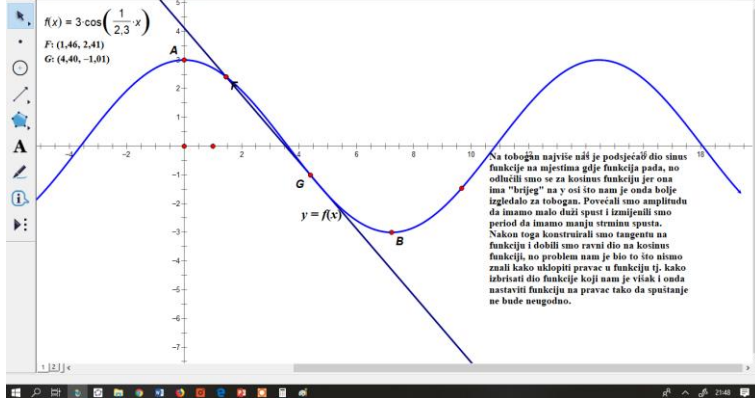
$$h(x) = 0,692987 \cdot x - 3,42843$$

Prije skoka moramo postići nekakav zalet, a to ćemo napraviti tako da uzmemo kosinusoidu i ograničimo je. Zatim za ravni dio uzmemo pravac i on mora biti "kao neka tangenta" u točki gdje kosinusoida prestaje. Uzmemo zadnju točku kosinusoide (u ovom slučaju $x_1 = 5\pi/4$) i uzmemo drugu točku koja je jako blizu te prve točke ($x_2 = 5\pi/4 \cdot 99/100$). Zatim možemo izračunati njihove y koordinate pomoću funkcije kosinus i kalkulatora. Tada imamo koordinate dvaju točaka i računamo parametre "k" i "l" u funkciji pravca $y = kx + l$.

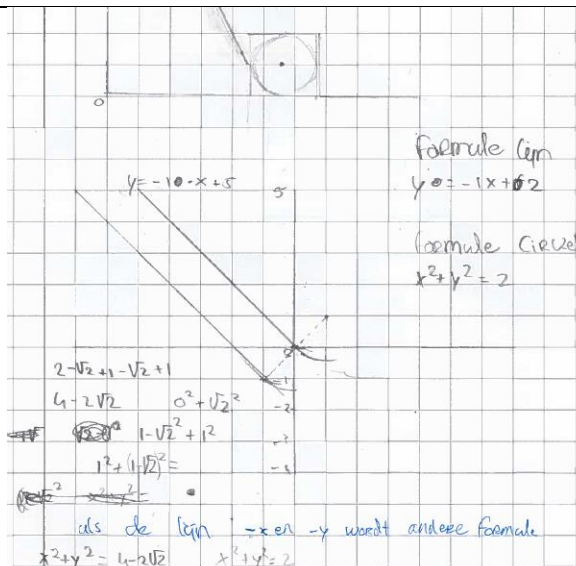
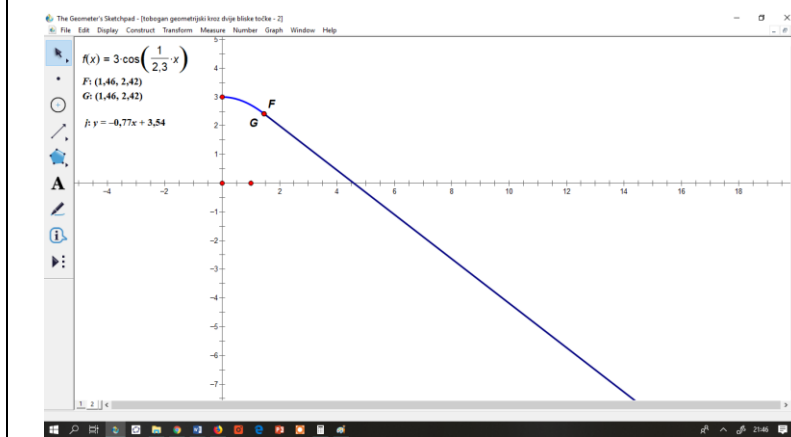
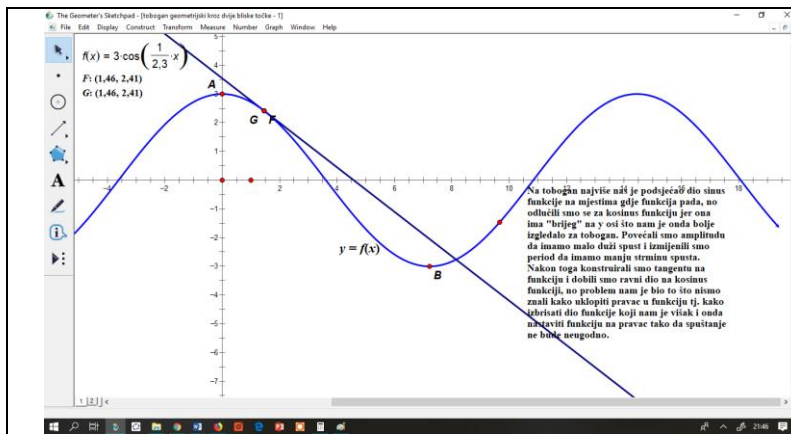
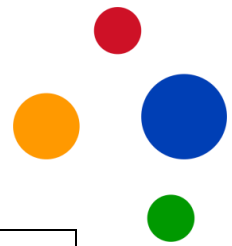


The Geometer's Sketchpad - [Ikkokovi dvije bliske točke - 1]

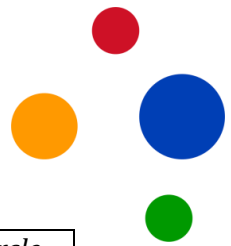
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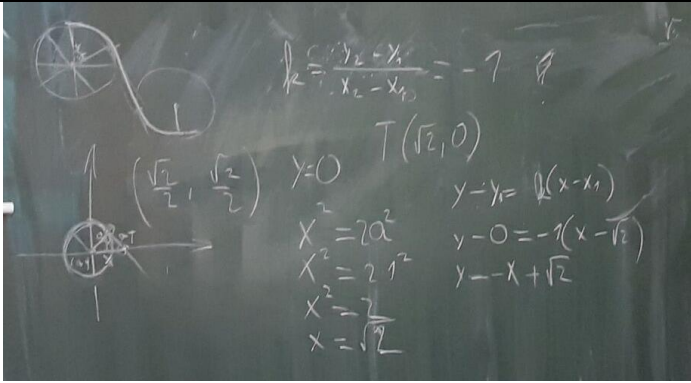
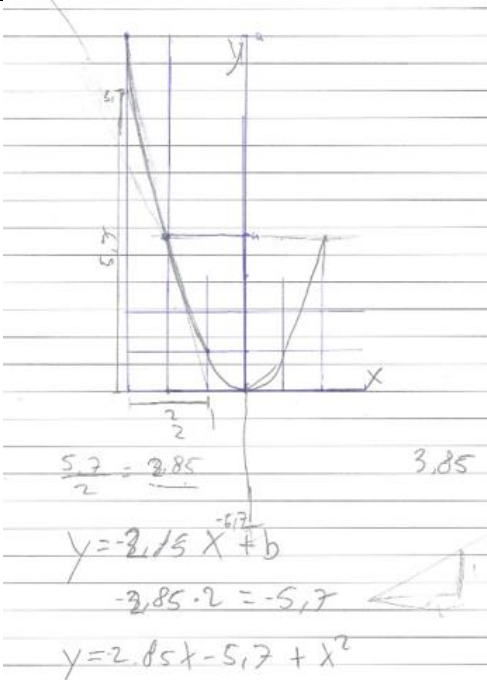
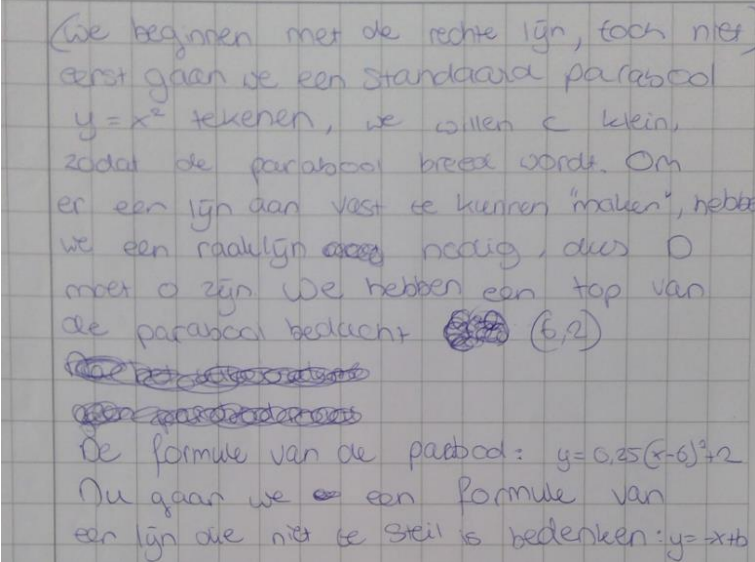


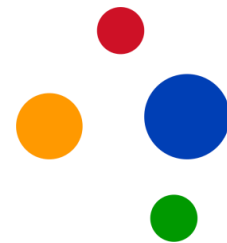
This group also used the secant lines approach (2). They read the equation from the screen.



This group tried the circle solution. They knew the tangent is perpendicular to the radius (so the validation is geometric: they use a theorem from planar geometry). However, they had difficulties making the circle and the line meet by vertical translation.



	<p>This group also has a circle solution. They know the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ on the unit circle and construct a line with slope -1 through the point.</p>
	<p>This group draws the graph of the parabola $y = x^2$. Then they try to fit a tangent line in one point in a way that seems to relate to linear approximation approach 3. They determine the slope of the line by measuring Δx and Δy and taking the quotient. The validation is visual.</p>
	<p>One group of students used the discriminant to decide whether there was one intersection point between their candidate tangent line and parabola. They fixed a parabola: $y = 0,25(x - 6)^2 + 2$ with the top $(6, 2)$, and tried to find the value of the parameter b in the equation for the line $y = -x + b$ using the discriminant.</p>



Evaluation tools

For a quick test of the acquired knowledge the teacher can pose questions:

1. The teacher draws an arbitrary curve and a line that is clearly not a tangent. (S)he asks whether the students think this shape is suitable for a train rail. The students explain why not.
2. Construct an equation for the tangent line to the unit circle at point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.
3. Approximate the slope of the parabola x^2 at point $(2, 4)$.

Suggestions for related topics

1. The notion of the *bounding line* and the difference from the tangent line.
2. Euclid's notion of a tangent – a line such that no other straight line could fall between it and the curve.
3. After learning about the limit of the difference quotient being the derivative function the students could calculate the derivative formula for a simple function, e. g. $f(x) = x^2, f'(x) = 2x$. The class can discuss what that means for the tangent lines of this function.

Rationale and RME perspectives on the scenario

Relevance and applicability

- Real-life: the problem is related to the everyday experience of the students. The objects they are asked to design are familiar to them and they know what the difference in shapes would result in. The task exploits their pre-knowledge of what it means for something to be smooth: namely their embodied experience of sliding down a smooth slide.
- World of work: The students learn to design and mathematize shapes they see in real life. They learn to connect the object with the mathematical representation. They simplify the shape of the object from a 3D-space to a curve. These skills are important in designing (for instance architecture) and modelling in professional contexts.
- Further study: The scenario is an introduction to derivatives and calculus.

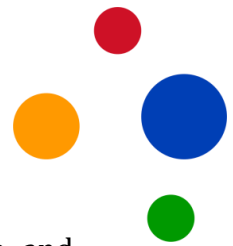
Inquiry skills

The students learn to simplify problems and apply mathematics to describe the part they are considering. They make hypotheses and test them. They generate examples and compare different solutions. They make decisions on which solution is better. They can argue for a better solution and communicate their arguments to others. They can extrapolate and generalize their results.

Potential for a sequence of lessons

This scenario can be a part of a series of lessons on derivatives. The pre-knowledge required of the students includes graphing functions, linear functions, and lines, examples of non-linear functions.

Further on in a series of lessons, the slide problem can be addressed with more formal methods, such as limits and derivatives. If the implicit derivation is introduced, the connection to the circle solution can be established. Apply implicit derivation to the circle

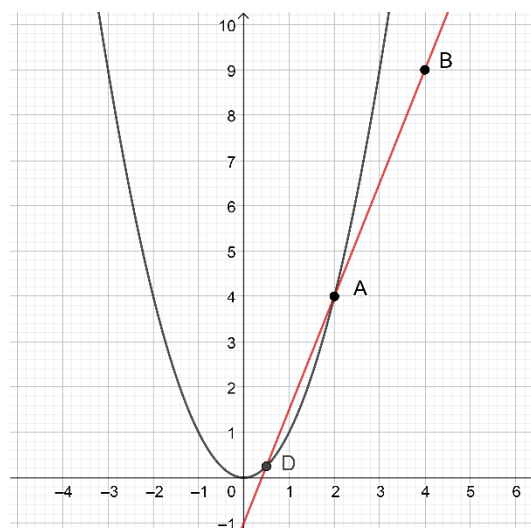


solution and compare it to the intuitive geometric solution of the current scenario, and the result obtained by explicit derivation. In this way, knowledge is validated by comparing different results produced by the students.

Rationale for the scenario

- *Horizontal mathematizing*: The problem is set in a rich context that is real to the student: all know what it means for a slide to be smooth or not. Students are invited to use mathematical language to model the situation: coordinate system, interpreting the 3-dimensional shape of the slide as a 2-dimensional curve; representing the curve with equations.
- *Vertical mathematizing*: In some cases, students develop these ideas by introducing parameters; and they discuss what to parametrize. Some apply other mathematical methods: like algebra (the discriminant) or difference quotients (slope of lines). From here there is a lot of potential for further mathematization. If students have used secant lines methods, then there is a natural transition to introducing the slope of a curve using the limit of the difference quotient. If they use algebraic methods and are focused on computing intersection points, there is an opportunity to discuss why there should be one intersection point (locally) and perhaps look at the multiplicity of the intersection points as a first step to compute the slope of a curve. If students used “zooming in” as a method of validation, then there is a natural connection to the local approximation approach to the slope of a curve.

Informal students’ models/solutions may be very diverse. The teacher has to find bridges from one approach (action/validation) to another as a means to move to a joint institutionalization. The *institutionalization* should be built on ideas that the students have come up with. For example, students may have been experimenting with the situation in GeoGebra:



They have varied the point B . The teacher points out that this corresponds to moving the point of intersection D . This can be a bridge towards discussing difference quotients and limits (in an informal way), and perhaps give the definition of derivative based on this.