MERIA Scenario “Slide”

Introduction to derivative

Target knowledge	Conceptual understanding of the slope of a curve as the slope of the tangent line.
Broader goals	Mathematical modelling of the slide using graphs of functions. Calculating the slope (the derivative of a function) by hand or using ICT. A meaningful introduction to calculus. Inquiry skills: experimenting with different graphs of functions on paper and using ICT, iterating a process to improve the solution, comparing different strategies, justifying the characteristics of the obtained solution. Interdisciplinary skills: students can connect their experience of smoothness of physical objects to mathematical terms of the tangent to a curve and the derivative of a function. The mathematical models may be used to produce 3D objects by printing on a 3D printer (ICT skills) or by production with other materials (crafts).
Prerequisite mathematical knowledge	Graphs and equations of a line and some nonlinear curves (circle, parabola or graph of the exponential function).
Grade	Age 16 - 18, grade 10 - 12 (whenever derivatives are introduced)
Time	60 - 90 minutes, two lessons
Required material	Paper, pencil, ICT - a tool for graphing functions, such as GeoGebra (the use of ICT is strictly speaking not needed but may greatly enhance the students’ experience).

Observations from implementation
The context of observations (grade, institution, country, etc.):

Problem:
Look at the pictures of a ski jump in-run and a children's slide. Both have a curved part at the bottom and/or the top and a straight part in the middle. Use mathematics to design such a shape. Focus on just one of the curved parts and the straight part in the middle. Remember, it is not nice to have a bumpy ride.
Introduce a coordinate system and find equations for *one* curved part and the line.

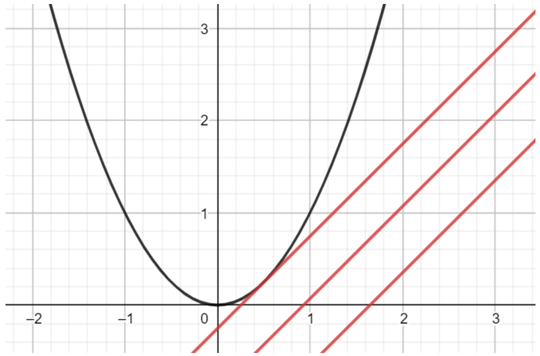
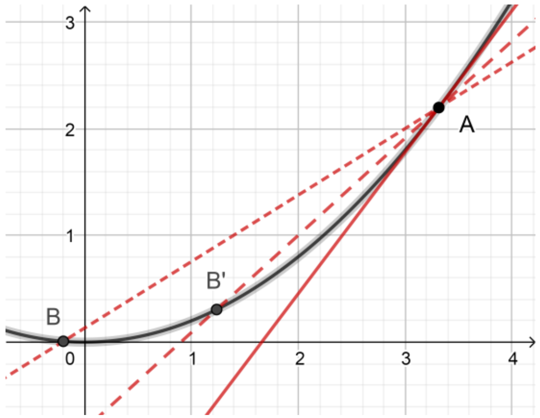
Note: For a longer lesson, with more modelling activity, omit this last sentence from the task description (see the module for extra lesson phases).

The Holmenkollen ski jump in Oslo, Norway. Photo taken by Mathias Stang and a children's slide.

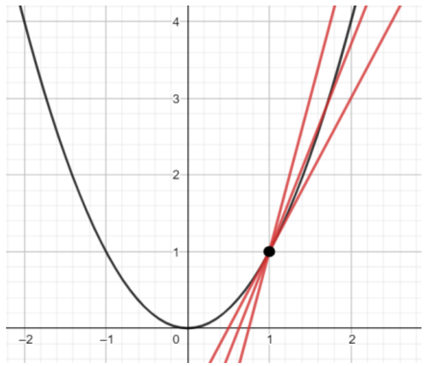
Phase	**Teacher's actions incl. instructions**	**Students' actions and reactions**	**Observations from implementation**
Devolution (didactical) 5 min	The teacher introduces the problem. The teacher points out that the students should design a smooth ride of a slide. The teacher makes sure that students focus on just one of the curved parts and the straight (linear) part in the middle.	Students sit down in groups of two or three. Students get excited!	
Action (adidactical) 20 min	Teacher registers the students' ideas, strategies, and findings. If students do not realize that the two parts should connect smoothly, the teacher should address this matter. If there are absolutely no ideas for the choice of the curved part after 10 minutes, the teacher reminds the students what the graphs of $y = x^2$ and/or $y = \cos x$ look like (not the circle), during a brief whole class (didactical) interruption. If students came up with the circle solution, one of the follow-up problems is: "What if you change the angle or what if you change the point in which the line and the circle meet? How does the equation for the line change?" After that, the teacher asks the	Student makes a sketch and introduces a coordinate system. Students' approaches can usually be described by one of the following categories: 1. Bounding line approach: they choose a free line and then move (translate and rotate) it until there seems to be just one intersection point in the area of focus. 2. Secant lines approach: they choose one point on the curve: the intended point of tangency; then another point on the curve, draw the line between the two points and move the second point closer to the first to obtain a smoother fit.	

	group of students to focus on the case where the curve is not a circle.	3. Linear approximation approach: Students choose one point on the curve, draw a line and then try to adjust the slope so that it fits best against the curve. Some may use a circle as a curve and the fact that a tangent is perpendicular to a radius. We call this the *circle solution*. See below for details on these (categories of) approaches in the section *Possible ways for students to realize target knowledge.*	
Formulation (adidactical) 15 min	The teacher asks the students to formulate their results. While they work on this, the teacher chooses groups with different approaches who will present their findings.	Students formulate their results within their group. For some groups a student presents their findings.	
Validation (didactical) 10 min	The teacher asks: “When do we know that the solution is good?” and “Is there a best solution?” If students have used visual validation only, the teachers could suggest algebraic or numerical approaches for validation.	They explain why some solution is good and whether one might be better than another. • Visual validation: Some will rely on their visual evaluation of the design; if it looks goods, then it is good. They may also zoom in on the curve. • Algebraic validation: The students may compute intersection point(s) algebraically and perhaps see it is locally unique.	

		• Numerical validation: Students can compute $\frac{\Delta y}{\Delta x}$ for two points on the curve and see if it is approximately the slope of their line. If the students have worked on a circle solution and computed the tangent line, they should be certain that they have a tangent and explain why (geometric and/or algebraic proof).	
Institutionalisation (didactical) 10 min	The teacher discusses the notion of the tangent line in the way that matches what the students came up with. The teacher can highlight one *or more* of the following viewpoints on the slope of the curve in a point: a) best local approximation follows visual validation b) locally unique bounding line - one intersection point follows algebraic validation c) classical definition using secant line and limits of difference quotients follows numerical validation If a circle solution comes up, a tangent to the circle and a tangent to other curves are discussed. Teacher recalls that the best solution for the circle is the tangent and that the students have actually approximated the tangent for the other curves.	Some may say something about the slope. Some may use the word “tangent” or the button in GeoGebra. Students listen and become interested in computing the best solution to the problem of arbitrary shapes and curves.	

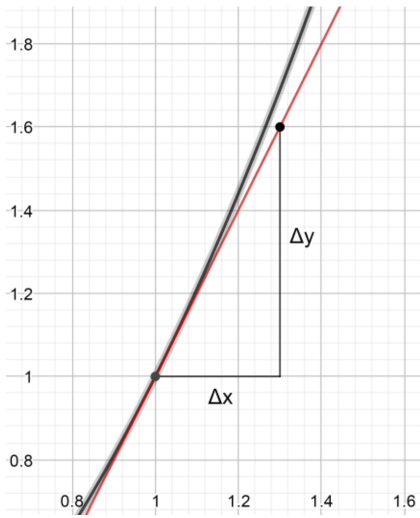
Possible ways for students to realize target knowledge	There are different options as to what the students do, for example: 1. Bounding line approach: Students choose for example $y = x^2$. Algebraic validation: from here consider the family of lines $$\mathrm{y} = \mathrm{x} + \mathrm{b}.$$ The bounding line is found by y-elimination: $$\mathrm{x}^2 = \mathrm{x} + \mathrm{b}.$$ This equation has a unique solution if the discriminant equals zero: $$1 + 4\mathrm{b} = 0.$$ So $\mathrm{b} = -\frac{1}{4}$ gives a smooth slide.  2. Secant line approach: Students fix one point on the curve, the intended point of tangency. Then they choose another point on the curve, draw the line between those two points and move the second point closer to the first to obtain a smoother fit. The closer you chose points the better approximation will be. This approach works best with ICT. 

3. Linear approximation approach:



For example, students work with $y = x^2$ and the point (1,1) where the curve ends and the line $\mathrm{y} = \mathrm{ax} + \mathrm{b}$ begins. Then they might guess $a > 1$ and try various values (of which $a = 2$ is correct). Trying means by drawing or graphing.

Describing the line by $y = ax + b$, they derive that $a + b = 1$.
So with each slope a, they can compute b.



Some may determine an approximation of a using two points on the drawn line using $\frac{\Delta y}{\Delta x}$.

Numerical example: students may find $a = \frac{\Delta y}{\Delta x} = \frac{0.6}{0.3} = 2$.
From $a + b = 1$ follows $b = -1$.

Validation is probably done visually, but it can be done numerically, possibly suggested by the teacher, because the method is similar. Now choose two points on the parabola and compute $\frac{\Delta y}{\Delta x}$; for example (1,1) and (1.1 , 1.21). Then

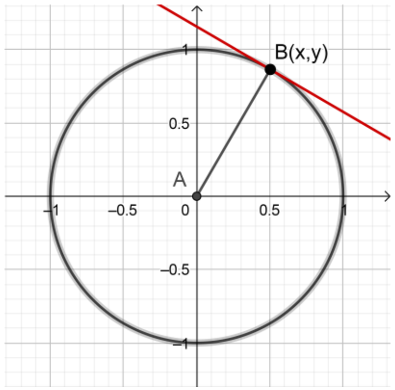
$$\frac{\Delta y}{\Delta x} = \frac{0.21}{0.1} = 2.1.$$

Pretty close!

Students can also validate by computing the intersection point of the line and the parabola (algebraic validation). If students are familiar with quadratic equations and the discriminant they can proceed by solving the system of equations: $y = x^2$, $y = ax + 1 - a$, and get $x^2 - ax + a - 1 = 0$. The equation will have exactly one solution if the discriminant is equal to 0:

$$a^2 - 4(a - 1) = 0 \;\Rightarrow\; a = 2.$$

4. Circle solution: Students choose a circle.
If the students have chosen a circle, they might choose
$$x^2 + y^2 = 1$$
and the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, that corresponds to the angle $\frac{\pi}{4}$. If they know that the radius of the circle is perpendicular to the tangent, they could determine $a = -1$. Then they could determine the equation of the tangent line.
If the teacher asks them to choose a different point (x, y), they could determine the slope a of the tangent line from the slope of the radial line through (x, y), which is $\frac{y}{x}$. So $a = -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$ (in general), but probably students will do this for one concrete point. This consideration may become a bit simpler if the students know and use vectors.



5. With ICT (GeoGebra or similar)
If the students use GeoGebra (or some other ICT), they would probably use similar steps and reasoning as without. The difference is that the ICT computes the equation of the line faster and draws an accurate representation of the chosen curve(s). With ICT the students can try more options in less time and can thus notice things, they would not with pen and paper. For example:
   - Some may find and use a button for the tangent line.
   - They might draw the curve and an arbitrary "good" line through a point on the curve and some other point. Then they zoom in and check whether it looks okay. They could move the second point to get a better fit. They could choose one possibility that they find is the best one and read the equation of the line by a "measuring tool".
   - Some students may begin by zooming in on a point until the graph of the curve looks straight. Then they could choose two points on it to compute an equation for it (or at least to draw a more or less tangent line).
   - They might try to see if their line has intersections with the curve (in this case it matters if they have drawn a half line or a line). Some might even let GeoGebra show the intersection points of the line and the curve and they could notice that when they change the slope of the line with a fixed point on the curve, they also change the (other) intersection with the curve (as pointed out in the Institutionalisation step). Because they immediately see the result, they could arrive at the hypothesis that the best solution is when points A and D coincide.