MERIA Scenario "Braking distance"

Quadratic relationship

Target knowledge	Braking distance is quadratically dependent on the initial speed.
Broader goals	Quadratic functions and their characterization by constant second derivative (second differences for quadratic sequences), or by constant decreasing or increasing first derivative (differences for quadratic sequences). Making calculations with different measuring units. Organizing data. Formulating functional relationship (writing the formula for function rule). Drawing graphs of (quadratic) functions on paper or using ICT. Inquiry skills: analysing data and looking for patterns in the tables, justifying findings (argumentation) during the presentations (the calculations dominate the process and students have to summarize their approach to others). Interdisciplinary skills: students have to work with variables from physics and make sense of the situation (bridging the two worlds of notations and procedures). Professional communication skills are emphasized in writing the report. Student also discuss responsibility of drivers and safety in traffic.
Prerequisite mathematical knowledge	Basic knowledge on functions, the relationship between constant speed and distance, average speed, conversion of km/h into m/s (and vice versa)
Grade	Age 16, grade 10 (whenever quadratic functions are introduced)
Time	90 minutes, two lessons
Required material	Handouts with tables to be filled, calculator, computer, graph paper

Observations from implementation

The context of observations (grade, institution, country, etc.):

Problem: In a city area with primary schools, parents complain about the set speed limit, considering it inadequate for the area with schoolchildren. A group of reckless drivers says that they do not need to worry because they brake in time. Now, you (the students) are asked to investigate how the braking distance relates to speed just before braking. Advise the mayor about the consequences of changing maximum speed. Underpin your advice with representations like tables and graphs.

Consider a car braking in such a way that the speed decreases by 10 km/h every 0.4 seconds. You can use the tables below to organize calculations, observe, and then justify your answer as you best can.

Phase	Teacher's actions incl. instructions	Students' actions and reactions	Observations from implementation
Devolution (didactical) 10 minutes	Teacher divides students into groups of three or four. The teacher poses the problem to students. (S)He makes sure that students understand the assumption of a constant decreasing speed during braking and discusses the idea of small time intervals where the movement can be approximated as the movement by the constant (average) speed. The teacher checks out that students understand the terms in the tables, the basic relationship between the speed, time and distance, how to convert km/h to m/s and the idea that 40 km/h could be replaced by other numbers. The teacher remarks to students they are free to use their own and different strategies. They are free to use any type of technology. Students are given a worksheet with the task. They are provided with a calculator (if students don´t have their own), computer and graph paper. Students are told that they have 20 minutes to investigate how speed and distance are changing and to make some conclusions about how they are related.	Students listen, talk about their ideas and answer the questions.	

Action (adidactical) 20 minutes	The teacher circulates, observes students working without interfering. In the case that many groups start a new table for every new initial speed the teacher might ask for a short plenary to ask how groups dealt with this issue. Probably, at least one of the groups realize that they can use previous calculations when trying to deduce the braking distance for other initial speeds and read from that table also the braking distance for lower initial speeds. This can be used as feedback for all other groups.	Students discuss in their group about strategies. They are completing tables using calculators or use ICT to graph points etc. They talk about precision, choosing different initial speed and similar issues. Members of the group might have different ideas and develop them individually. Students might use calculations, graphs or physics laws to come to conclusions: - braking distance is not changing with a constant rate, - the relation between the initial speed and distance is not linear, - as the initial speed increases, braking distance also increases, but not proportionally. Some students might notice that 2nd differences are (approximately) constant and use recursion method for calculations.	
Formulation (didactical) 10 minutes	The teacher goes to each group and asks them to present shortly what they have found. (S)He might ask questions and discuss their ideas, particularly if they have stuck.	Students present their work shortly and ask questions.	

	The teacher asks groups with different strategies (within the group) to focus on one strategy that they will use for generalizing and presenting their ideas (due to lack of time). The teacher reminds students that the goal of the activity is to find out how the braking distance relates to speed just before braking to be able to do predictions and to give proper advice to the mayor. Therefore, students are asked to prepare advice to the mayor about the consequences of changing maximum speed and underpin their advice with representations like tables and graphs.		
Action and formulation (adidactical) 20 minutes	The teacher is observing.	Students are trying to generalize their calculations and observations. Some of them might change the strategy for generalizing or approach to the problem. Students are preparing advice to the mayor.	
Validation (didactical) 25 minutes	The teacher asks students to present and compare their strategies.	Students present their work, listen, ask questions and discuss other strategies and solutions.	

Institutionalisation (didactical) 5 minutes	The teacher highlights the mathematical differences and similarities in the student's strategies, explains why some strategies will not provide proof but might be convincing from the graph and a formula that might be produced by technology, that the relationship is quadratic. The teacher introduces quadratic function.	Students listen and connect their solutions with a general quadratic function.	

Possible ways for students to approach target knowledge	Students will fill the given table with data (v, d).

	Time (seconds)	**Change in speed during braking (km/h)**	**Average speed (km/h)**	**Average speed (m/s)**	**Time interval Δt (s)**	**Distance traveled Δd (m)**
	$t = 0$ to $t = 0.4$	$v = 40$ to $v = 30$	35	$\frac{175}{18}$	0.4	$\frac{35}{9}$
	$t = 0.4$ to $t = 0.8$	$v = 30$ to $v = 20$	25	$\frac{125}{18}$	0.4	$\frac{25}{9}$
	$t = 0.8$ to $t = 1.2$	$v = 20$ to $v = 10$	15	$\frac{25}{6}$	0.4	$\frac{15}{9}$
	$t = 1.2$ to $t = 1.6$	$v = 10$ to $v = 0$	5	$\frac{25}{18}$	0.4	$\frac{5}{9}$
Distance traveled after braking (m)						$\frac{80}{9}$

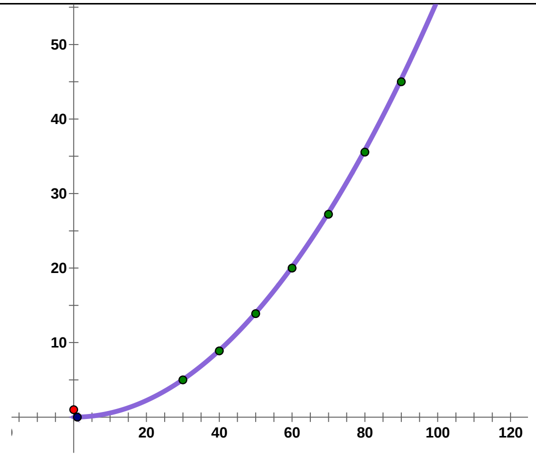
Speed just before braking (km/h)	30	40	50	60	70	80	90	100	110
Braking distance (m)	5	$\frac{80}{9}$	$\frac{125}{9}$	20	$\frac{245}{9}$	$\frac{320}{9}$	45	$\frac{500}{9}$	$\frac{605}{9}$

Or with decimals, for instance:

Speed just before braking (km/h)	30	40	50	60	70	80	90	100	110
Braking distance (m)	5	8.89	13.89	20	27.22	35.56	45	55.56	67.22

By looking at the data they can conclude:

- The braking distance is longer when the speed is higher.
- The relationship between speed and braking distance is not linear ($\frac{\Delta d}{\Delta v}$ is not constant).
- If the speed doubled, the distance is increased four times. If the speed increases three times, the distance increases nine times.
- Students can draw points (v, d) and conclude that the relationship might be quadratic. They can write a quadratic function
$$d = av^2 + bv + c$$
and determine unknown coefficients a, b, c using data from the table and solving system of linear equations. They will get an approximation. This strategy will not provide proof that the relationship is quadratic.



- After the conclusion that the relationship might be quadratic, students can use ICT to find quadratic regression. They will get an approximation. This strategy will not provide proof that the relationship is quadratic.
- From the data in tables students can generalize:

$$d_{40} = 5 \cdot \frac{5}{18} \cdot 0.4 + 15 \cdot \frac{5}{18} \cdot 0.4 + 25 \cdot \frac{5}{18} \cdot 0.4 + 35 \cdot \frac{5}{18} \cdot 0.4$$

$$d_{40} = \frac{5}{9}(1 + 3 + 5 + 7) = \frac{5}{9} \cdot 16 = \frac{80}{9} \approx 8.89$$

$$d_{50} = d_{40} + 45 \cdot \frac{5}{18} \cdot 0.4$$

$$d_{50} = \frac{5}{9}(1 + 3 + 5 + 7 + 9) = \frac{5}{9} \cdot 25 = \frac{125}{9} \approx 13.89$$

$$d_{60} = d_{50} + 55 \cdot \frac{5}{18} \cdot 0.4$$

$$d_{60} = \frac{5}{9}(1 + 3 + 5 + 7 + 9 + 11) = \frac{5}{9} \cdot 36 = 20$$

$$d_{v_0} = \frac{5}{9}\big(1 + 3 + \cdots + (2n - 1)\big) = \frac{5}{9} \cdot n^2$$

An important conclusion is that if we observe the braking distance, we look for the moment when the speed is equal to 0; so many times we will subtract 10 of v_0 until we get 0:

$$v_0 - 10n = 0 \Longrightarrow n = \frac{v_0}{10}$$

$$d_{v_0} = \frac{5}{9} \cdot \left(\frac{v_0}{10}\right)^2 = \frac{1}{180} {v_0}^2 \approx 0.0056\, {v_0}^2$$

In this formula we substitute v_0 in km/h and get the distance in metres.

- Students can use calculators and write data in tables as decimal numbers. The results will not be exact and it is not easy to recognize patterns.

- Students can use information that the speed decreases by 10 km/h every 0.4 seconds. They might calculate that the speed decreases by 25km/h every second, or by $6.9\dot{4}$ m/s every second, which means that acceleration is $a = 6.9\dot{4}$ m/s². Then they use formulas from physics:

$$v = v_0 - at, d = v_0 t - \frac{a}{2} t^2 .$$

They use an important conclusion: if we observe the braking distance, we look for the moment when the speed is equal to zero. From the first formula ($v = 0$) they calculate time $t = \frac{v_0}{a}$ and substitute in the second to get

$$d = \frac{v_0^2}{2a} = \frac{9v_0^2}{125} = \frac{v_0^2}{13.\dot{8}} = 0.072 v_0^2.$$

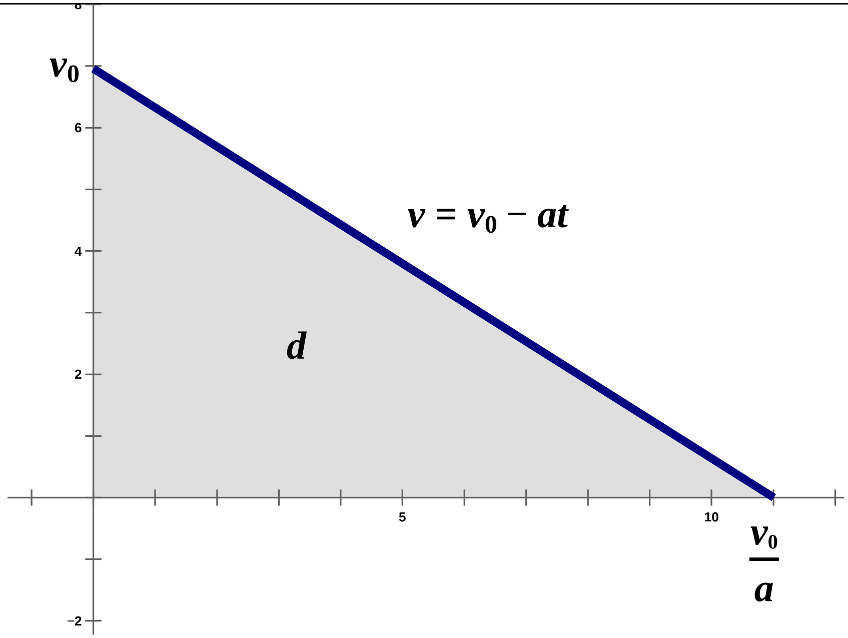
In this formula we substitute v_0 in m/s to get the distance in metres.
- If students calculate acceleration in km/h² they will get:
$a = 90000$ km/h², substitute v_0 in km/h and get distance in kilometres

$$d = \frac{v_0^2}{180000}, \text{ or in metres } d = \frac{v_0^2}{180}.$$

- Students can draw a v-t graph and calculate the distance as the area under the graph:

$$d = \frac{1}{2} \cdot \frac{v_0}{a} \cdot v_0 = \frac{v_0^2}{2a} = 0.072 v_0^2.$$

In this formula we substitute v_0 in m/s.


v_0
6
4
2
−2
$v = v_0 - at$
d
5
10
$\frac{v_0}{a}$

	Time (seconds)	**Change in speed during braking (km/h)**	**Average speed (km/h)**	**Average speed (m/s)**	**Time interval Δt (s)**	**Distance traveled Δd (m)**
	$t = 0$ to $t = 0.4$	$v = 40$ to $v = 30$	35			
Distance traveled after braking (m)						

Speed just before braking (km/h)	40							
Braking distance (m)								