MERIA PRACTICAL GUIDE TO INQUIRY BASED MATHEMATICS TEACHING

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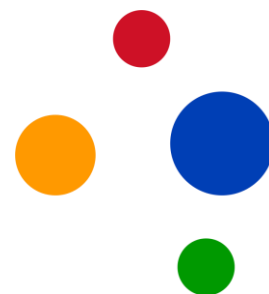
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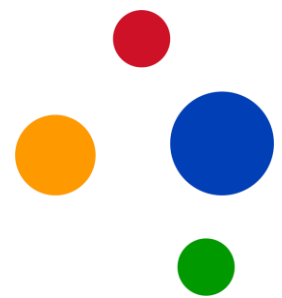*(this page has intentionally been left blank)*

Table of Contents

Introduction

This booklet presents the theoretical basis for the MERIA project, and is especially intended to support the design of scenarios and modules in the project, as well as the analysis and evaluation of their effects.

MERIA aims to further the use of relevant, interesting and applicable mathematical activities in secondary school classrooms. The main hypothesis of the project is that such activities engage students in more serious mathematical work than solving exercises with predefined methods. In fact, the "exercise paradigm" in many everyday practices of teaching mathematics (including upper secondary and even university courses) may be a main factor that shapes the common students' impression of mathematics as uninteresting (tedious routine work), irrelevant (at least to them) and useless (except to pass an exam). The alternative proposed and pursued in this project can be roughly characterized as *inquiry based mathematics teaching*, where exercises are replaced by "inquiry activities" of various types. Designing such activities, testing them in practice and disseminating them to teachers, are thus our main tasks.

The project aims to be based on serious and visionary research on how to realize the aforementioned tasks. This is the reason why in this volume we have gathered a presentation of important approaches and ideas from the research literature. The volume is structured in four chapters:
- Chapter 1 presents the general idea of "inquiry" in mathematics education, both from a historical point of view, and in terms of how it may be defined at present (in general and relatively broad terms).
- Chapter 2 provides general strategies for implementing inquiry as a students' activity in classrooms.
- Chapter 3 and 4 present two more precise - and well established - research programmes for the design of inquiry based mathematics teaching:
  - The Theory of Didactical Situation in mathematics, which strives to put students in "research like situations" (akin to mathematicians): consisting of *action, hypothesis formulation*, and their *validation/proof.*
  - Realistic Mathematics Education, in which mathematical notions are built up from students' work with problems in contexts that are "real" to them, through the "mathematisation" of those contexts.

References to the literature are provided throughout for those who wish to pursue a given point at greater depth than it was possible in the present volume. The appendix provides an outline of some of the most important references for this project. At the end of the handbook, a glossary of the most important special terms, employed in the text, is also provided.
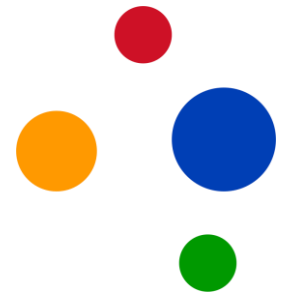
1. What is Inquiry Based Mathematics Teaching?

*Inquiry* can loosely be defined as "investigating a problem". Here, the word "investigate" implies that the efforts to solve the problem are relatively autonomous: not directed by others and not following a prescribed routine method. *Inquiry Based Mathematics Teaching* (IBMT) is then a teaching approach that allows students to be engaged in an activity which leads them to adapt their existing, or construct new, mathematical knowledge. Such a teaching is supposed to foster students' understanding of the meaning and foundations of secondary mathematics. It is particularly fruitful when it comes from students' own activity and effort.

In this chapter, the emergence and detailed meanings of IBMT will be presented. In fact, mathematics education research has produced different and well established conceptualisations of the above, rough idea – that is, methods for teaching mathematics in a way that students question, explore, hypothesize and reason about mathematical ideas. Nonetheless, the general term IBMT is of quite recent origin.

To distinguish between IBMT and other ways of teaching mathematics we especially need to clarify what is meant by "a problem", in what way it is different from a task or an exercise, and why solving a problem is not the same as solving an exercise. Finally, we will discuss the role of students' questioning the problem and related content knowledge. Research indicates that it is crucial that students tackle the problem or situation by themselves, since this can drive them to formulate hypotheses, explore and experiment with their knowledge, as well as to formulate solutions based on their actions.

Before we turn to describe these components of IBMT we will give a short account of how and why IBMT has recently appeared as an overarching approach to the development of mathematics teaching. Certainly, MERIA is not the first European initiative to promote IBMT. During the last decade or so, the European Union has financed several large-scale projects with the aim of developing, implementing and assessing "inquiry-based science teaching" at different levels of the educational systems (Artigue & Baptist, 2012; Mass & Artigue, 2013; Ropohl, Rönnebeck, Bernholt, & Köller, 2016). Most of these projects pursued also mathematics together with science. In fact, the notion of "inquiry based teaching" is more natural and prominent in science education than in mathematics education. In mathematics education, more or less similar ideas and approaches have been developed under the labels of problem solving, mathematical experimentation or mathematical modelling, etc. However, in science education, as well as in mathematics education, there exist different approaches how to develop this kind of teaching. This handbook covers two such approaches in mathematics education in more details (Chapter 3 and 4). According to the Fibonacci project, inquiry in science education often draws on sense experience (Artigue & et al., 2012, p. 9). There are many science notions related to sense experiences, like speed, time, light, force, pH value, changing seasons, etc. Those
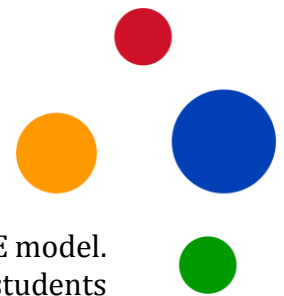
experiences can be further studied in cyclic processes, e.g. the so-called 5E model. The 5E model refers to phases of inquiry based science teaching where students are supposed to **e**ngage in, to **e**xplore, to **e**xplain, to **e**laborate and to **e**valuate the knowledge or ideas to be developed during an inquiry process (Bass, Contant & Carin, 2009, p 91). In the 5E model, the sense experience of force or time, ecosystems, or chemical reactions from everyday life, can serve as an outset or starting point which engages students in more systematic inquiry into a phenomena or causal relations. These inquiry processes might lead students to construct knowledge of laws in natural sciences.

By contrast, one can argue that mathematical knowledge is often built from a more theoretical basis. Certainly, inductive reasoning based on "experiments" can be set up in many cases, such as for number patterns or specific examples of a more general principle. But as argued by Artigue and Baptist (2012), the cumulative nature of mathematics represents a challenge with respect to adopting the notion of inquiry directly from natural sciences (Artigue & Baptist, 2012). In science, a hypothesis (by a researcher or a student) is validated by experiments, while in mathematics, the ultimate validation requires a proof based on deductive reasoning.

This booklet will provide you with two different approaches to Inquiry Based Mathematics Teaching (IBMT). One of them presents examples of how students' experiences can serve as an outset of an inquiry process. It is called Realistic Mathematics Education (RME) and was initially developed by Hans Freudenthal (Freudenthal, 1991). The other approach is the Theory of Didactical Situations (TDS) initially developed by Guy Brousseau (Brousseau, 1997). TDS is based on the idea that students construct new knowledge when they solve a problem while adapting to what is called a didactical milieu. We will go into further details with RME and TDS in Chapter 3 and 4. In this chapter, we will present core notions of IBMT, to elucidate its origins, justifications (*why* it is important to pursue) and constraints (what challenges it may encounter).

Origins of IBMT

More than a century ago, the first formulations of the idea that teaching in general should be related to students' experiences and should be focused on students' activities were put into writing. The educational researcher John Dewey is often associated with the phrase "learning by doing". He argued that teaching should rather revolve around students' activity and the ways students can gain knowledge from it (Dewey, 1902). Dewey (1938) emphasised the potential importance of inquiry and its role in learning and teaching – especially in the field of science. To a large extent, he regarded mathematics as a tool or a language to order complex data and to carry out systematic treatment of outcomes of inquiry processes – for instance the outcomes of students' actions when experimenting with laws of physics or biological systems. Although Dewey did not provide explicit proposals on how to create inquiry based mathematics teaching, his ideas have been pursued later on by several mathematics educators.
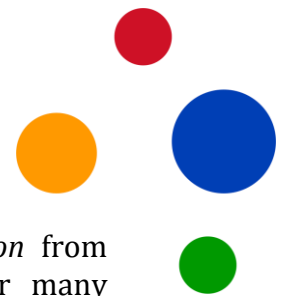
What Dewey opposed was a long tradition of *knowledge transmission* from teachers to students, which is as old as the discipline itself. For many mathematicians to teach meant to repeat, and to make students recite a given text or imitate the teacher's action while solving mathematical problems. This also goes for the more practical and elementary sides of mathematics, related to calculation techniques. Even today, much mathematics teaching is based on repeating demonstrated techniques, and training them to perfection through endless sequences of similar calculations. In many schools, a common "template" for mathematics teaching consists in teacher's presenting of some technique (e.g. a formula, a rule, a method, etc.), after which he provides his students with a few "typical" examples on how to employ the new piece of knowledge in solving mathematical tasks of a certain type, and finally he provides the students with very similar tasks so they can practice what the teacher did (Schoenfeld, 1988). An example could be when students are presented with the definition of the second degree polynomials and how to find its roots from the equation

$$ax^2 + bx + c = 0.$$

Students are then provided with the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Next, the teacher shows the students how to find the roots of a polynomial e.g. $2x^2 + 2x - 12$ by using the provided formula. More examples might be given before the students are asked to solve a series of similar exercises. In this way students imitate the teacher's activity, while they may ignore the meaning of finding the roots, as well as the justifications of the method. Asking students to solve equations where the polynomial has one or no real roots could, on the other hand, create potentials for students' inquiry of the meaning of polynomial roots and equation's solutions.

Routine exercises simply require students to imitate a teacher, which they may often do without seeing or constructing any rationale or meaning of tasks and techniques used to solve them (Schoenfeld, 1988). In fact, students may well - over time - get to consider mathematics as rather meaningless set of techniques which have to be acquired by imitative training. This kind of teaching fails to give students experiences of many important sides of mathematics, such as: solving complex problems, building up coherent structures of knowledge, conjecturing and proving, experimenting with special cases, etc.

One can argue that to transmit knowledge and to teach students how to solve standardised tasks has proved to be sufficient for many former students in high schools throughout the last century. However, today larger and more diverse groups of students enter upper secondary education in many countries. To teach them requires more elaborate approaches underpinned by research in mathematics education. The research field of mathematics education has emerged through centuries, beginning with teachers who have shared reflections on their teaching and developed teaching techniques based on their own
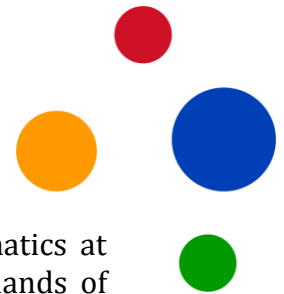
experiences (Kilpatrick, 2014). Today, students need to learn mathematics at deeper levels of understandingthan in the past in order to meet demands of society. In former times, it was common that people left school before upper secondary education to enter the work market. This required merely practical mathematical skills such as techniques to calculate and repeat established procedures. Today many professions and higher education require that students graduate from high school with knowledge and competences regarding basic calculus, statistics, the notion of function etc. This growing number of students at upper secondary level, some with very little motivation for learning mathematics, represents a specific challenge for mathematics teaching. These students may simply not be able to receive transmitted knowledge as easily as earlier generations, which is a reason why more inquiry based approaches are needed. To understand the ways how students develop mathematical knowledge is an ongoing research interest in mathematics education. Mathematics educator Mogens Niss formulates the rationale of this interest as:

> *If we understood the possible paths of learning mathematics, and the obstacles that may block these paths, for ordinary students, we would gain a better understanding of what mathematical knowledge, insight, and ability are (and not are), of how they are generated, stored, and activated, and hence of how they may be promoted* (Niss, 1999, p. 4).

Throughout the 20th century, different approaches to this interest have been developed, but a persisting one is to let teaching be inspired by the ways in which professional mathematicians think, learn and develop mathematics.

Early in the 20th century mathematicians Fehr, Laisant, Hadamard and other collected systematic accounts of how they and their colleagues develop new mathematical knowledge - in order to characterize the research activity and to let researchers serve as inspiration for students' engagement in learning mathematics (Kilpatrick, 2014). This idea was also seen in the first reform movements with respect to high school curriculum, during which the German mathematician Felix Klein (beginning of the 20th century) introduced a reform programme for teacher education promoting practical instructions, development of spatial intuition and a functional approach to mathematics (Kilpatrick, 2008). Klein played a key role in the early development of the research area of mathematics education, and especially for the relation between research in mathematics, the teaching of mathematics, and research in mathematics education. In various and often indirect ways, his ideas continue to influence mathematics teaching at the secondary level. Next wave of reforms, which can be related to the idea of IBMT, is the introduction of problem solving in mathematics education in the 1980s. We consider it in the next section as an underlying idea of inquiry in mathematics.

Characteristics of inquiry processes

In this section we present an overview of ideas and concepts which govern IBMT developments throughout the last century. A core notion is that of a *problem*.
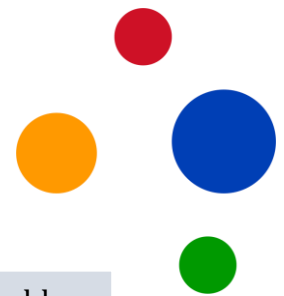
In IBMT *a problem is* more than a certain task, exercise or an activity. A problem is open in the sense that it requires students to engage in experimenting, hypothesizing about possible solutions, communicating hypotheses and possible solution strategies, and maybe pose further questions to be studied as a part of a process of its solving.

An example of a problem could be the following:

"Consider an arbitrary triangle with side lengths *a, b* and *c*. If the sides are all equally enlarged, how much bigger is the area of the enlarged triangle compared to the initial one?"

Depending on the context in which the problem is posed, it provides students with different opportunities for engagement in actions of characterizing a problem and finding its solution. There exist different solution strategies to this problem depending on students' prior knowledge on triangles, measures of sides, angles and areas, similar triangles and trigonometric ratios. Students can tackle the idea of enlargement and experiment with additive and multiplicative enlargement. They can actually create a large number of triangles, enlarge them, collect results and formulate hypotheses regarding the enlargement of the area. Further, students can look at special cases (such as right angled triangles) and algebraically deduce a hypothesis regarding how much bigger the enlarged area will be. Different solution strategies can further be compared, discussed and even validated or tested on new triangles. In the context of IBMT, it is an asset that students can pursue different and personal ideas, compare, connect and evaluate those in order to construct more robust knowledge. This means that students know more than to calculate the area of a triangle. They know how to combine new knowledge with other relevant domains in order to solve open problems. The knowledge constructed with this problem relates to symmetries and mappings between geometrical shapes.

In mathematics, the actions and experiences, which Dewey suggests learning should arise from, will most often be motivated by attempts to solve some problem. The exposed triangle enlargement problem is an example. Problems can vary in nature, in their origin, level of difficulty, number of possible strategies or solutions, etc.; they may also have different potential for generating mathematical curiosity or creativity among students. This is important as a catalyst for students' questioning and experimenting with the content knowledge.

Other examples from school practice can cover dynamic uses of computer programs or modelling situations outside of mathematics. While working in a Computer Algebra System or a Dynamic Geometry Software with a graphical environment, a student might draw a graph of a linear function given by $f(x) = ax + b$. Here a student has an opportunity to move or tilt the graph up or down. The student might investigate what happens with the graph while changing the coefficients. This problem could be driven by curiosity and assist students to
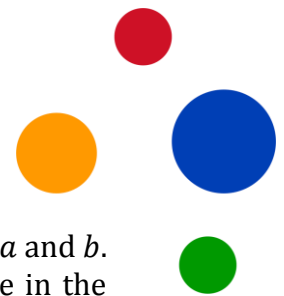
develop knowledge about the graphical interpretation of the coefficients a and b. ICT technology and different software in general play an important role in the development and support of IBMT (e.g. see Artigue & Baptiste, 2012, p. 10).

Students can encounter many other questions where the need for knowledge is elicited, such as "Can all natural numbers be written as a product of prime numbers? Can they be written as a sum of primes?" or "How can I describe the acceleration of my bicycle on my way to school when I have measurements of the speed at certain moments of time?" These questions or problems require from students to develop new knowledge. The problems can rise from purely mathematical issues, as well as they can be initiated by experiences or actions in real world. Problems, problem formulation and problem solving are the core components of the inquiry process in mathematics teaching and play a significant role in the mathematics education literature. We will now give a brief overview of how these notions have been addressed throughout the last century in the context of mathematics teaching.

Problem solving as a way to learn

Just as important element in origin of IBMT as problem posing is the problem solving activity. Although *problem solving* has become the core element in mathematics curricula of many countries from 1980s, it was not a new notion in the literature at that time. In 1945 George Pólya published his book "How to solve it?", which is considered a classical reference in problem solving approaches to mathematics education (Artigue & Blomhøj, 2013, p. 802). The book describes the problem solving as an activity in which mathematicians engage when doing research. Emphasis was put on the role of problems and heuristic competences needed to solve these problems. Heuristic competences draw on content knowledge and strategies needed to address non-routine problems. The triangle problem is such a problem, a non-routine problem for school context. It can be solved by employing content knowledge such as the area of an arbitrary triangle, but Sit also requires that students develop knowledge about similar triangles while combining known strategies and pieces of knowledge in new ways. In his work, Pólya suggests that students use the strategies such as to find a counter example, to sketch a situation e.g. with a graph, to consider special cases, to guess and check, to prove by contradiction etc. In the triangle problem a good starting strategy can be to consider special cases, such as concrete triangles or right angled triangles. Knowledge used can be a definition, a rule, a method etc. These are all familiar components of mathematical activity at university level. However, Pólya's work does not systematically address how to realise these activities at all levels of the educational system when teaching mathematics.

Schoenfeld is one of the researchers who were engaged in pursuing realization of Pólya's ideas in mathematics teaching systematically. He criticized the 1980s uses of Pólya's work as being trivialized (1992, p. 352), in particular, not sufficiently emphasising the crucial element of students' development of heuristic competences. Schoenfeld argues that before getting students to engage in problem solving activities one needs to distinguish between problems and
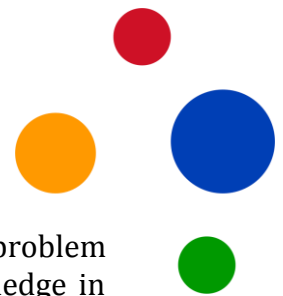
exercises. Exercises can be solved by *known solution strategies*, whereas problem solving activities require developing or combining methods and knowledge in new ways. Schoenfeld identifies important elements in this process which require that students can draw on resources. Resources are mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand; intuitions and informal knowledge regarding the domain, facts, algorithmic procedures, "routine" non-algorithmic procedures, understandings (propositional knowledge) about the agreed-upon rules for working in the domain. Hence, this is what students need to learn to draw upon during problem solving processes: their previous acquired knowledge, competences and skills and to combine them with intuition and preliminary hypothesis of an answer. To realise this, students draw on their heuristics, which include "strategies and techniques for making progress on unfamiliar or nonstandard problems; rules of thumb for effective problem solving, including: drawing figures, introducing suitable notation, exploring related problems, reformulating problems; working backwards, testing and verification procedures" (Schoenfeld, 1985, p. 15). These heuristics have similarities with Dewey's description of the role of mathematics in inquiry processes, but they also go beyond that. According to Artigue and Blomhøj (2013), they share many characteristics with approaches of inquiry based science teaching, such as questioning, hypothesizing, experimenting systematically, collaborating, communicating, representing the problem in different ways etc., all with purpose of students' developing new knowledge. It is also important to notice that the heuristic competences include an explorative and curiosity driven attitude towards mathematical activity.

This all could be reflected in the triangle enlargement problem with its possible solution strategies. The problem can be solved by experimenting with concrete materials (creating the triangles), considering different reformulations of the problem (enlarging with additive or multiplicative strategies) or the problem can be addressed from an abstract point of view by e.g. addressing a special case, for instance, a triangle with a right angle.

> In IBMT *problem solving* is the activity in which students are expected to engage. It includes the students' use of previously developed knowledge, intuition, vague ideas, and hypotheses to explore and understand the problem. Through experimentation and new ways of combining their knowledge, including knowledge developed during the exploration, students construct new knowledge, which is to be evaluated through further experimentations. Students' mathematical creativity and curiosity drives the problem solving process, and are also further developed by engaging in problem solving.

However, it is still somewhat unclear how to teach students in this way, when and why to apply each element. Teachers must develop problems where students need to act as mathematical inquirers but teachers themselves are supposed to refrain from telling the students what to do. However, concerns in teaching are not only to provide students with the opportunity to gain experience with problems that are of more open character than ordinary exercises, as exemplified
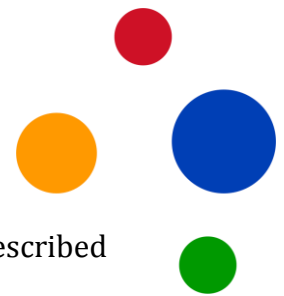
with the triangle problem, but also to go successfully through the whole described process of solving a problem.

The amount of guidance in problem based teaching

Already in 1938 Dewey advocated that students' learning processes should be anchored in students' interactions with a problem. Preferably this interaction happens in a dialectics between familiar and unknown situations, where the already acquired knowledge of the students should guide the study of the unknown. For example, based on what students already know, they can formulate hypothesis and approach the problem in a systematic way during the inquiry process. In the triangle problem the students can systematically explore the relation between the enlargement of the triangle and the enlargement of areas. Students can formulate precise hypotheses regarding this relation from the special case of the right angled triangle. The hypothesis can be evaluated by creating arbitrary triangles and calculating the areas of the initial and the enlarged triangle. Based on experiences from the actions, students construct their own knowledge regarding the studied problem. In the triangle problem, students have potential to construct knowledge regarding similar triangles, a general formula for the area $A = \frac{1}{2}ab\sin C$, where C is the angle between the two sides a and b, and understanding of the trigonometric ratios. Therefore, to design inquiry based teaching, a teacher must create scenarios which help them to understand in which phase students draw on already acquired knowledge in the study of a problem, where a hypothesis is created and where it is tested, where new knowledge can be constructed or formulated based on the (generalisations) of the students' actions. In this sense the teacher serves as a facilitator in creating and guiding the students in their knowledge construction (Godino et al., 2015). Teachers' role should be as the experienced co-researcher guiding the younger members of the research community rather than play the role of the person with all the answers (Artigue & Baptist, 2012).

> To scaffold students' work with a problem in IBMT relates to the problem formulation. The formulation should enable students to develop a multitude of strategies, depending on what knowledge they have already learned. It should further promote students' exploration and experimentation with the problem, leading them to construct new knowledge. In this process, the teacher should guide the students - not by providing them with answers, but as an experienced co-researcher who poses questions and thus drives the inquiry process.

Today it is generally agreed upon that real problem solving ads to the learning outcomes of mathematics teaching: "one gains *more* from solving a problem than getting to know the answer obtained" (Bosch & Winsløw, 2016). This "more" relates to the heuristics and uses of resources mentioned above. It could appear elusive, but is still recognizable when encountered. Previous studies indicate that good problem solving teaching is about creating appropriate *relations* between specific students and specific tasks (Schoenfeld, 1992, p. 353). Therefore, during the recent decades, research focus was on characterizations of problems which
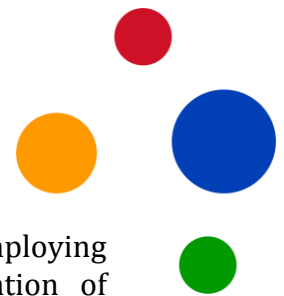
are suitable – they should contain rich potential for students' employing heuristics and resources. Furthermore, focus was also on exploration of instruction and guidance principles that are supposed to be efficient in making students realise their full potentials. In the 1980s, suggestions with respect to instructions varied from guided student practice to practice that requires students to articulate the processes as a kind of *meta-reflection* on their own practice. Such meta-reflections could be that the students need to translate the problem into mathematics, for example in the problem with modelling the students' acceleration during the bike ride to school. Furthermore, meta-reflection is needed in the problem which considers the graph of a linear function in the coordinate system and requires students to collect concrete data from a series of examples in order to formulate hypotheses regarding coefficients. Instruction principles also deal with challenges and concerns of teachers when to interfere with students' activities or refrain from providing the students with answers, or how to elicit the optimal strategies (Schoenfeld, 1992, p. 354). If the teacher suggests that students should consider a special case, plot data points in order to carry out linear regression or to sketch the problem as a graph or a geometric shape, some students will interpret this as the only possible way how to solve the problem. Not because they are convinced that the strategy solves the problem, but rather because the teacher says so. Therefore, it is difficult to guide or scaffold students' work without indicating an answer. In the triangle problem, it is a non-trivial task for a teacher to guide students if they insist only on working with concrete examples. A question such as if their hypothesis holds in general might lead to new approaches to the problem, but it might also be too overwhelming and in reality not serve as guidance. This is the general challenge of scaffolding inquiry processes in a teaching context. Students need to be provided with a restricted field in which to pursue their inquiry, but if the guidance is too directed, or the restrictions too many, the potential of the teaching design is ruined. Then the students cannot construct knowledge based on their actions and experiences. Therefore, a teacher cannot tell the students explicitly what to do. At the same time, the teacher must create the needs of students to act in a way which might lead them to reach the intended learning goals. Hence scaffolding should be thought of as something else than providing examples, strategies and posing too direct and closed questions.

> The challenge of scaffolding students' inquiry processes: with too much direction, there is no real inquiry, and the learning potential is ruined; but with too little direction, students get stuck and disengage from solving the problem. Giving the "right" amount of direction is a delicate act of balance.

The role of students' questioning when dealing with problems

More recent studies suggest problem posing as an approach to engage students in problem solving. The pedagogical idea of problem posing may be as old as the idea of problem solving. Ellerton (2013) refers to Einstein and Infeld who claimed that the formulation of a problem is more essential than providing the answer, and that it is a more demanding task (Ellerton, 2013, p. 88). This might be an
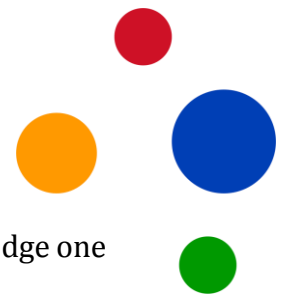
exaggeration, but it emphasises the importance of questioning the knowledge one intends to study further or learn more about.

The emergence of problem posing as an approach to mathematics teaching is linked to the renewed interest in problem solving in the 1980s. Ellerton's study of engaging pre-service mathematics teachers in problem posing activities suggests that the problem posing might get a dominant role in mathematics curricula (Ellerton, 2013, p. 90). Reasons for this claim is that problem posing nurtures students' development of the heuristics and resources involved in problem solving and IBMT, or, as it is formulated by Singer, Ellerton & Cai "Problem posing improves students' problem-solving skills, attitudes, and confidence in mathematics, and contributes to a broader understanding of mathematical concepts and the development of mathematical thinking" (Singer, Ellerton & Cai, 2013, p. 2). According to Artigue and Blomhøj (2013), but also Hiebert et al. (1996), problem posing as a student activity may also support Dewey's idea of reflective inquiry, which implies that students should be allowed and even encouraged to problematize the knowledge to be taught. Moreover, this means that students should be encouraged to wonder about the implications of the problems they are faced with during IBMT. In the triangle example, it is reasonable for students to question what is meant by "to enlarge equally" the sides of a triangle. However, it is important that the students themselves provide an answer, for instance by experimenting with both additive and multiplicative strategies. This supports the students' autonomous construction of knowledge, in particular a discovery that the additive strategy does not lead to similar triangles. This might further lead students to a question if areas can be compared only in the cases where the initial and the enlarged triangle are similar. Further, the triangle problem might lead students to question how to find the height of an arbitrary triangle by knowing only the side lengths as a, b and c, etc.

> A good problem has an openness which leads students to wonder, delimit and pose questions on the content knowledge involved. The questioning is crucial as a driver of the inquiry process and should lead students to answer their own questions and hypotheses.

In terms of guidance and scaffolding, a number of design ideas have been developed to realize problem posing as an activity in school mathematics. These ideas span from delivering information to students and ask them to pose problems which can be solved using the received information; ask them to solve certain problems and afterwards formulate similar problems, or describe some phenomena to students and ask them to pose problems regarding the described phenomena (Bosch & Winsløw, 2016). However, these design ideas are still somehow off the mark if students' activity should reflect the activities of research mathematicians, where new questions rise from the researchers' interactions with the knowledge domain at stake, for instance through the study of other researchers' work or through personal or collective problem solving activities. These activities might lead to a wondering or a curiosity and then to formulation of new problems, which drive the research further. According to Kilpatrick this is
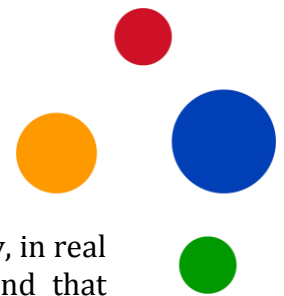
not a characteristic of research alone. He claims that even more generally, in real life, most problems are formulated by a person who solves them and that mathematics teaching should be more akin to real life in this respect (Kilpatrick, 1987, p. 124). The explicit role of problem posing in the different approaches to IBMT vary, but it is a shared trait to seek to make students wonder, make them curios and investigate assumed relations and formulate hypothesis for further inquiry.

Where do the problems come from?

Along with the problem posing literature, other theories on mathematics teaching have addressed the role of a problem, problem solving and problem posing in different ways. Solving non-routine problems is a corner stone in several approaches to mathematics education, which was initiated and further developed in the 1970s and onwards: The Theory of Didactical Situations (TDS, Brousseau, 1997) and Realistic Mathematics Education (RME, Freudenthal, 1991). A shared idea in TDS and RME is that students should be provided non-routine problems, which they solve through the development of new knowledge. In TDS, this is supposed to happen through students' adaptions to what is called the milieu of the teaching situation (Brousseau, 1997). In RME the development of knowledge happens when students mathematize the phenomena addressed in the problem. RME distinguishes between two aspects for this process: vertical and horizontal mathematisation (Freudenthal, 1991). Both theories share the idea that a teacher provides students with the initial problem, but students are supposed to act and formulate ideas related to the problem solving. These activities might lead students to implicitly or explicitly question the knowledge at stake. The two approaches are the turning points of the activities in MERIA and elaborated presentations of the two theories will be given in later sections of this booklet.

However, there are other theoretical approaches which address IBMT as well. Mathematical Competence Theory (Niss et al, 2002) can be said to cover heuristic competence even if the theory operates with eight competences, none of which is called heuristic. Especially, the so-called problem solving competence and modelling competence share traits of what has been described above as the crucial role of heuristics in inquiry based activities.

One can also argue more generally that *mathematical modelling activities* support the development of the problem solving skills and attitudes. From the perspective of Mathematical Competence Theory, mathematical modelling activities can be described as cyclic motions forth and back in certain phases of the modelling cycle (Blomhøj, 2004; Blum & Leiss, 2006). The modelling cycles can serve as guidance for teachers; by being aware of the phases which are parts of modelling activities, teachers could follow their realization by students in teaching situations. However, studies employing the modelling cycles for analyses of students modelling activities indicate that this is not always the case. Modelling problems which carry the potentials of realising all phases may still not always realize these potentials in teaching contexts (Blum & Borromero Ferri, 2007). Mathematical modelling from the perspective of Mathematical Competence Theory and RME
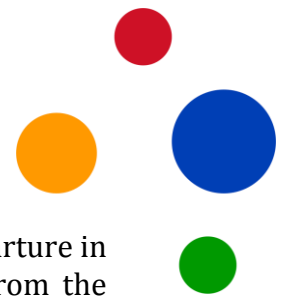
approach share the idea that the inquiry processes takes its point of departure in "realistic" problems. This also aligns with Dewey's idea on teaching from the beginning of the 20th century. (There are also) Other modelling approaches, such as Modelling Eliciting Activities (Doerr & Ärlebäck, 2015), advocate more open approaches to the inquiry processes and how they can be supported by students' already acquired knowledge in intra as well as extra-curriculum mathematical knowledge domains.

The same can be said about the Anthropological Theory of the Didactics. ATD provides a general model of mathematical knowledge seen as a human activity of study of types of problems. It is organized by mathematical praxeologies which consist of a practical block (types of problems and techniques) and knowledge block (technology and theory). It considers a teacher as the director of the didactical process. In this approach the inquiry process is initiated by an open problem posed by a teacher. The nature of this problem can be purely mathematical or from the real life. Students' explicit formulation of question is supposed to drive the inquiry process (Chevallard, 2015). However, teaching and learning are not seen as isolated but take place in a complex process of didactical transposition. This process also involves some didactical restrictions coming from different institutions involved (society, mathematical community, educational system, school, classroom) which reduce the autonomy of teachers. ATD also proposes ways to transform conditions and constraints of schools and disciplines. This ambitious form of IBMT will not, however, be pursued further in this handbook.

In the Japanese tradition (Nohda, 2000), open ended approach with a large variety of students' answers to the same problem is seen as a force which focuses the attention of students (and teachers) on mathematical argumentation and communication. Different strategies developed by students can be shared in the whole class so that the students construct more coherent knowledge regarding the problem.

In different theoretical approaches, we thus find different ideas on how to construct and find the problems, which can initiate rich inquiry processes among students. In the area of mathematical modelling, the initial problems are often provided by teachers and in most cases they address issues from real life scenarios. But a common feature of IBMT, which goes back to Dewey, is the idea of students learning from interacting with one or more problems.
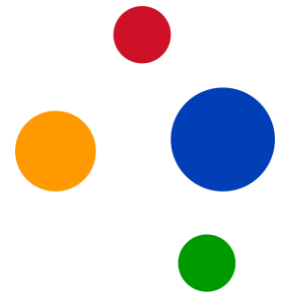
To summarize the ideas presented so far, we can say that IBMT is described as any teaching activity that aims for students to engage in processes of inquiry in mathematics - which means that they construct, question and explore concepts and notions, by acting or learning to act as an inquirer with a mathematical problem, and thus develop a certain mathematical knowledge.

What has been done to promote IBMT?

Researchers in mathematics education seek to promote IBMT through national and research projects across countries. In many countries IBMT is at some degree included in the mathematics curriculum, although different formulations of its characteristics are seen to be used and different theoretical approaches are reflected. Projects financed by the European Union support and study the implementation of IBMT in the educational systems within member countries. In 2007, an EU report on trends in pedagogy in the European educational systems reported upon "the alarming decline in young people's interest for key science studies and mathematics in Europe" (Rocard et al., 2007). In the same report, it was proposed that "the reversal of school science-teaching pedagogy from mainly deductive to inquiry-based methods provides the means to increase interest in science". Similarly, in the USA, the document "Principles and standards for school mathematics" emphasises as the goal for secondary mathematics teaching to "teach students to solve non-routine problems by offering them the potentials of developing knowledge and tools for solving such problems" (NCTM, 2000). This indicates national interests worldwide in promoting IBMT in all classrooms. However, national accounts regarding the promotion of IBMT still point to many challenges for implementation. Reports from France and England argue that politicians have many good intentions, but may not be aware of what it actually takes to change the classroom practices, how to transform teaching from transmission of knowledge to IBMT (Burkhardt & Bell, 2007; Artigue & Houdement, 2007). The EU project PRIMAS investigated these challenges and recommends that teachers should be provided by opportunities to implement inquiry based mathematics teaching. Furthermore, it suggests that all projects and course activities taken to promote IBMT should be adjusted to local conditions. However, teachers need structures that support them and that promote to support each other in the implementation of the new initiatives (García, 2013). Similarly, the EU project MASCIL investigated challenges how to implement inquiry-based teaching and how to connect mathematics and science education to the world of work. It promotes a holistic approach in offering the support by "carrying out a variety of activities, including development of high-quality, innovative materials and running professional development courses" by IBL-trained teachers as multipliers.

We now turn to the question of realising IBMT, including the challenges this involves.

2. How to pursue IBMT?

Introduction

Inquiry Based Mathematics Teaching (IBMT) has been promoted in several EU-projects to better prepare students for a dynamic, knowledge-based society. Knowledge of facts and isolated basic skills alone are not sufficient for the 21st century. Students need to develop problem solving skills and the ability to acquire knowledge by themselves. Consequently, competencies that are becoming increasingly important are the ability to deal with missing information, to be mathematically creative in new knowledge areas, to collaborate in problem solving situations and to communicate (mathematical) results. Mathematics education has a responsibility in developing 21st century skills (Rocard et al., 2007).

Attention for these 21st century skills is not new. Similar competences can be recognized in initiatives that intend to measure and promote the mathematical literacy of students. Mathematical literacy is defined by PISA as:

> … students' capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals in recognizing the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens. (OECD, 2016a, p. 25)

Also, in the USA, current common core standards address competences that transcend procedural fluency (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010). In these standards explicit attention is given to the importance of developing competences like problem solving, reasoning, communicating and representing.

These lists of new competences all share the need for more flexible skills. The question is how these skills can be developed in mathematics classrooms? A way to address these skills is with IBMT. In IBMT processes like hypothesizing, planning investigations, experimenting systematically and evaluating results are used to create classroom practices. This chapter will address the role of tasks and teaching strategies for IBMT. In addition, experiences with IBMT in various European countries will be described and discussed.

Tasks that foster IBMT

Addressing inquiry-based skills can be difficult with traditional textbook tasks. These tasks often offer exactly the information needed to solve the task and they are mostly structured in such a way that students hardly need to think about the solving procedure. For pursuing IBMT it is important to create classroom practices in which inquiry-related processes can be addressed. Not necessarily all these processes from the circle of inquiry need to have explicit attention with each task for the students, but the task should provide opportunities to learn about at least one of these inquiry-related processes in mathematics. Unstructured tasks
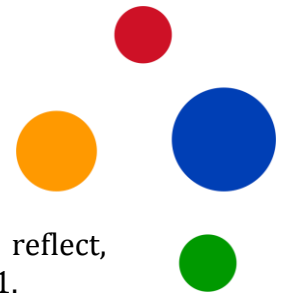
can provide such opportunities for students to inquire, critically reflect, collaborate and communicate results; an example is presented in Figure 1.

Structured version	Unstructured version
A patient is ill. A doctor prescribes a medicine for this patient and advises to take a daily dose of 1500 mg. After taking the dose an average of 25% of the drug leaves the body by secretion during a day. The rest of the drug stays in the blood of the patient. • How much mg of the drug is in the blood of the patient after one day? • Finish the table.	A patient is ill. A doctor prescribes a medicine for this patient and advises to take a daily dose of 1500 mg. After taking the dose an average of 25% of the drug leaves the body by secretion during a day. The rest of the drug stays in the blood of the patient. Investigation • Use calculations to investigate how the amount of the drug (in mg) changes when someone starts taking the drug in a daily dose of 1500 mg with for instance three times 500 mg. • Are the consequences of skipping a day and/or of taking a double dose really so dramatic? • Can each amount of drug in the blood be reached? Explain your answer. Product Design a flyer for patients with answers to the above questions. Include graphs and/or tables to illustrate the progress of the drug level over several days.

Inside the Structured version cell:

Day	Mg of drug in blood
0	0
1	1125
2	
3	

• Explain why you can calculate the amount of drug for the next day with the formula: new_amount = (old_amount + 1500) * 0,75
• After how many days has the patient more than 4 g medicine in the blood? And after how many days 5 g?
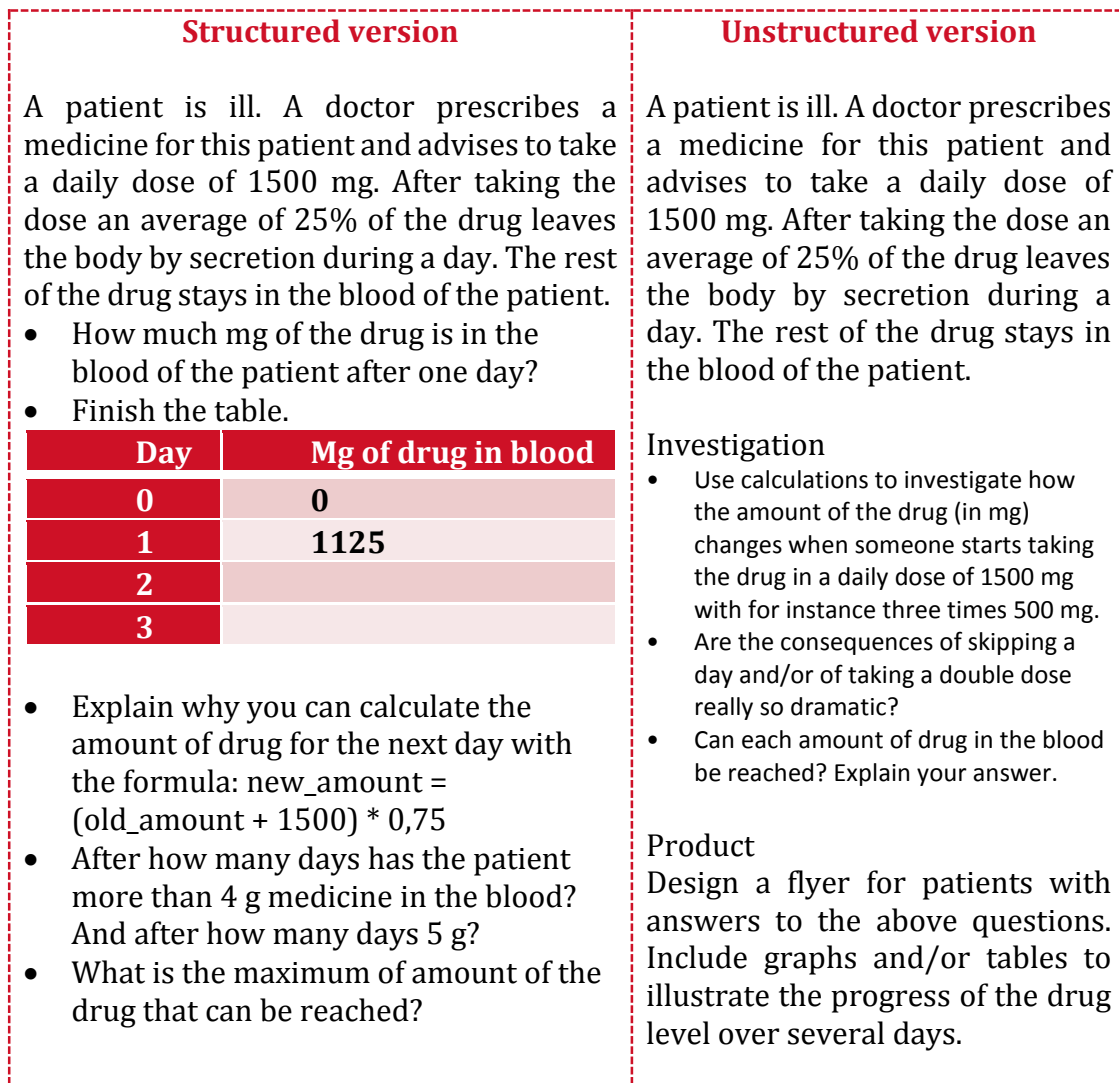• What is the maximum of amount of the drug that can be reached?

Figure 1: Two versions of a task (Doorman, Jonker & Wijers, 2016, p. 25)

When using the unstructured version of the task in a mathematics classroom it is the teacher's responsibility to keep students' focus on the mathematical aspects of the problem (or to what extent it is allowed to move from mathematics to e.g. biology). This example shows that 'open' tasks might connect mathematics with other sciences and shows that mathematics is applicable. However, 'opening up problems' does not necessarily imply situating mathematical concepts in non-mathematical contexts. Also pure mathematical tasks can be unstructured and presented as an investigation. An important goal of unstructured problems is to put students in the active role and to foster their agency in mathematical problem solving.

In IBMT learning is driven by unstructured tasks that give rise to multiple-solution strategies. The strategies of the students, their interpretations of the
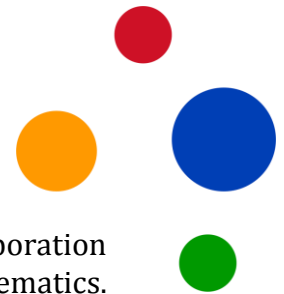
problem, their assumptions, calculations, representations and collaboration provide opportunities to reflect upon inquiry-related processes in mathematics. During this process teachers are proactive. They support and encourage students who are struggling and extend those that are succeeding through the use of carefully chosen strategic questions. They value students' contributions - including mistakes, and scaffold learning using students' reasoning and experience. This requires in the classroom a shared sense of purpose, i.e. creating mathematics together, and ownership.

Important to stress is that in daily practice not everything needs to be or can be changed towards inquiry-based teaching. The role of inquiry in daily teaching practices is one of the ingredients of good education. The pursuit of IBMT can be fostered by supporting teachers in extending their teaching repertoire towards topics like addressing processes of inquiry in daily practice, developing resources for IBMT, being aware of ways to learn concepts through IBMT, supporting collaborative work and measuring progress and evaluating students' IBMT-related competencies.

Teaching strategies for IBMT

PRIMAS[1] was an EU-project that aimed at supporting teachers to collaboratively investigate IBMT pedagogies by designing and implementing modules for professional development. These modules included activities to connect inquiry-related teaching methods with existing practices, innovative classroom activities, illustrated with classroom videos, and sample lesson plans. The modules were expected to enable teacher educators and teachers to be challenged and to act reflectively in new ways (Swan et al., 2013).

Moving towards an IBMT approach raises many pedagogical issues for teachers. For instance: How can I encourage my students to ask and pursue their own questions? How can I help students to follow up these questions in profitable ways? How can I teach students to work cooperatively and to learn from each other? How can I manage all these new activities within the constraints of my daily responsibilities? These questions gave rise to the following topics that were elaborated in the PRIMAS modules to promote inquiry in daily classroom practice:
1. organizing student-led inquiry;
2. helping students to tackle unstructured problems;
3. promoting concept development through inquiry;
4. asking questions that promote reasoning (and include all students);
5. supporting collaborative work;
6. using self- and peer assessment to promote learning.

Transformations of a textbook task

We illustrate some examples of alternatives ways to use the textbook task that is presented in Figure 1. The structured version of the task presents a context, states the problem and exactly the information needed to solve it. The task asks for using

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[1] http://www.primas-project.eu/
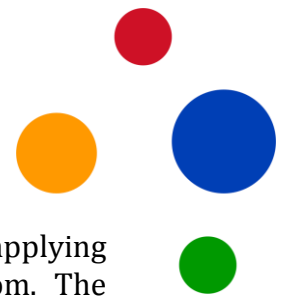
a formula and the context can be neglected. The task does not support applying or learning to apply mathematics outside the mathematics classroom. The unstructured version seems to provide opportunities for students to inquire the situation, to be mathematical creative, to collaborate, to critically reflect on the findings and to communicate their results. However, the unstructured version of the task also has the risk that students feel lost and don't know what to do, or that parts of the task will ask so much time from the students that they are unable to reach a reasonable result within the timeframe of the lesson. To prevent this to happen, the teacher has a role in structuring the lesson. Consequently, the unstructured version of the task needs a structured lesson plan to scaffold the students' inquiry.

Lesson 1
10 minutes: create groups & introduce the problem and the working plan and distribute the task
10 minutes: students work in groups on the task
10 minutes: discuss with the whole class whether all groups have an idea how to start and how to proceed. Exchange strategies and make sure that everybody has an idea what is expected.
15 minutes: students work on the task, finish calculations and prepare the building blocks for their flyer.

Lesson 2
20 minutes: students finish their flyer
20 minutes: presentations of a few examples
10 minutes: reflection on the task (and positioning it in further work)

Figure 2: A structured lesson plan for an unstructured task (see Figure 1).

It should be noted that this lesson plan requires pedagogical skills to manage the classroom process. The teacher needs to change a few times during the lesson from whole-class discussions to group work.

Another option to involve students in the inquiry-process - with more structure - is to cut the task into pieces. You could show only the introductory text and ask: What is the main problem? Is any further information needed to tackle the problem? What strategy could you pursue to find answers?

A patient is ill. A doctor prescribes a medicine for this patient and advises to take a daily dose of 1500 mg. After taking the dose an average of 25% of the drug leaves the body by secretion during a day. The rest of the drug stays in the blood of the patient.

Figure 3: The situation of a task. What could be the main problem?

After students formulated the problem by themselves, they can be given the structured textbook version. Probably the order of the questions now makes
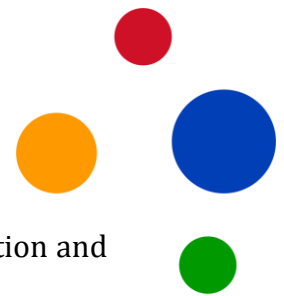
more sense to them, since they had the opportunity to think of the situation and possible strategies by themselves (Ainley et al., 2009).

The last option we use as an illustration of what can be done with textbook tasks is taking all sub questions and presenting them a different order, or as pieces of a jigsaw and asking the students to find the original textbook order.

The task presented in Figure 4 gives students the opportunity to reflect on the structure of their textbooks. In many cases the tasks have a similar kind of structure, that reflect a sensible strategy to inquire the problem and to find the answer to the main question, but students are almost never asked to reflect on this strategy and to describe its characteristics (perform one calculation, systematically collect more data in a table, describe the calculation-process with a formula, draw a graph, and use formula and graph to solve the main problem).

A patient is ill. A doctor prescribes a medicine for this patient and advises to take a daily dose of 1500 mg. After taking the dose an average of 25% of the drug leaves the body by secretion during a day. The rest of the drug stays in the blood of the patient.

- What is the maximum of amount of the drug that can be reached?

- Explain why you can calculate the amount of drug for the next day with the formula: new_amount = (old_amount + 1500) * 0,75

- Finish the table.

| Day	Mg of drug in blood
0	0
1	1125
2	
3	

- After how many days has the patient more than 4 g medicine in the blood? And after how many days 5 g?

- How much mg of the drug is in the blood of the patient after one day?

Figure 4: The order of questions is mixed up. What was the original textbook order?

More teaching strategies for IBMT

We have presented three alternative ways of using a textbook task and transforming it in order to address inquiry in the mathematics lesson. They have in common that the teacher needs to be able to arrange classroom discussions and
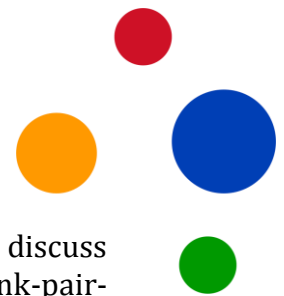
provide students with thinking time. Another teaching strategy to discuss processes of inquiry with the whole class, to involve everybody, is the think-pair-share strategy. The main idea is to let students think 2 minutes for themselves about a problem and note what they think, followed by 2 minutes for comparing thoughts with their neighbors, and finally 2 minutes for sharing the results with the whole class. This strategy provides all students thinking time and gives the teacher the opportunity to involve everybody in the discussion.

In addition to the above tasks, one could also think of tasks that encourage students to challenge hypotheses (e.g. Figure 5). You could provide students with a set of statements that are related to the topic that you are teaching and let them decide whether these statements are always, sometimes or never true. If they think a statement is *always* or *never* true, they are expected to explain how they can be sure. If they think it is *sometimes* true, they need to describe when it is true and when it is not.

Pay rise	Right angles
Max gets a pay rise of 30%. Jim gets a pay rise of 25%. So Max gets the bigger pay rise.	A pentagon has fewer right angles than a rectangle.
Birthdays	**Bigger fractions**
In a class of ten students, the probability of two students being born on the same day of the week is one.	If you add the same number to the top and bottom of a fraction, the fraction gets bigger in value.

Figure 5: Statements that are always, sometimes or never true.

This task invites students to decide on the validity of statements and give explanations for their decisions. Probably, explanations will involve generating examples and counterexamples to support or refute statements. In addition, students can be asked to add conditions or otherwise revise the statements so that they become 'always true'. This kind of activity is very powerful. The statements may be prepared to encourage students to confront and discuss common misconceptions or errors. The task of the teacher is to prompt students to offer justifications, examples, and counterexamples. This task offers students opportunities to discover the role of examples in mathematical inquiry.

These examples of tasks for mathematics show the importance of carefully chosen resources for pursuing IBMT in the mathematics classroom.
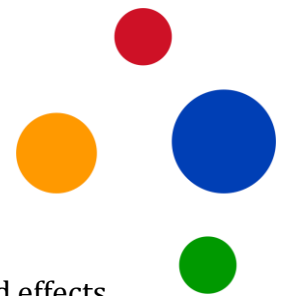
Experiences with implementing IBMT

There is some empirical evidence from different studies on the quality and effects of IBMT. The effects of IBMT include benefits for motivation, for the development of beliefs about mathematics as well as for understanding the relevance of mathematics for life and society (Bruder and Prescott, 2013; Blanchard et al., 2010; Furtak et al., 2012; Hattie, 2009; Minner). However, some experts caution that this type of instruction can only improve learning if it is carefully designed and well-structured (Hofstein and Lunetta, 2004; Woolnough, 1991).

Guided inquiry, in comparison with structured inquiry and open inquiry, was shown to be the most effective method of implementing inquiry in the classroom in combination with closed tasks that support learning of procedures and basic skills (Bruder and Prescott, 2013).

How do we know when any teaching is effective? Many of the pressures on teachers today arise from the different expectations of student learning, which are often not clearly formulated. If we want students who are able to understand mathematics, enjoy mathematics and have an ability to work through problems and to draw conclusions, then IBMT with guidance could be thought successful. However, if our aim is for students to be able to achieve high marks in standardized knowledge-based tests then IBMT is sometimes less successful.

Results of the internal qualitative evaluation of PRIMAS give a rich flavour of challenges and opportunities that teachers confront when experimenting with IBMT pedagogies (Maass, 2013). Most of the PRIMAS teachers consider IBMT as a student-centered approach which involves self-directed but guided exploration, asking questions, making discoveries, and testing assumptions in search of new understanding.

> *Inquiry is about giving priority to students to generate explanations and engage in critical discourse instead of not requiring any thinking at all [...] in solving complex problems, students apply their knowledge to new real world problems, and engage in critical discourse with others about models, solutions and documentation.* (Teacher from Cyprus)

However, the implementation of attention for processes of inquiry within class is seen as a challenging but fruitful opportunity to design lessons in a different way. Teachers highlight the benefits that inquiry-based learning has for their students.

> *From our own experience, we know the value of having found something by ourselves, instead of having simply been taught the solution. When teaching inquiry-based learning, students really learn an approach, they then have more keys for understanding.* (Teacher from Switzerland)

The teachers also emphasize the positive impact of inquiry-oriented processes on students reasoning.

> *I have realized that there was an impact on students' inductive reasoning. I was impressed by the ability of some students to make robust conclusions, and support them using mathematical evidence, in the form of models [...] students' oral participation has been dramatically increased [...] especially the use of correct*
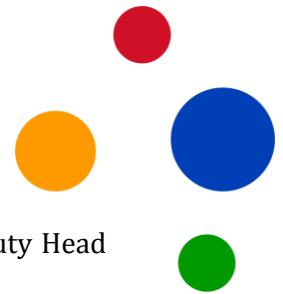
mathematical terminology, something that is not easy at this age. (Deputy Head from Cyprus)

Teachers indicate that in their own practice, they loved explaining concepts and procedures – and so do some of their students. However, lessons in which students have to struggle with open questions and problem-solving tasks appear to be also effective for discussing solution strategies.

> *I liked teacher-centered teaching and I think that students still do like it. But they won't learn that much since they won't have to solve a problem themselves. They will get the problem, the procedure of solution and the solution itself at the end.* (Teacher from Germany)

In this context, the importance of students exchanging with their classmates is highlighted by the teachers:

> *I discovered how important it is to get students to use that kind of opportunity (dialogue) to start figuring out what they know and what they might be able to learn from others. And then students might notice that they end up with an answer they may have thought they did not have.* (Teacher from Norway)

An example from the Netherlands

Working with IBMT involves a change of roles, both for teachers and for students. Teachers take up the role of a learning-facilitator and students will be given a very active part. For example, one of the teachers took a mathematics textbook exercise from an algebra chapter and devised an unstructured version of the task. The original task consisted of a series of pyramids in which students had to add or multiply adjacent cells to determine the content of the cell above them. In some cases they had to reason 'backwards' to find the content of cells in a lower region (see Figure 6). With her adaptation of the task she wanted to get access to what students were capable of and what they thought to be easy or difficult. She first presented one of such pyramids and asked her students to try to find out how this pyramid was constructed and whether they could find the values of the empty cells.
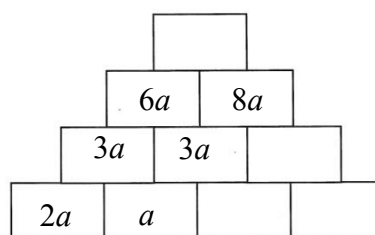

Figure 6. Pyramid as designed by the teacher

After five minutes and some discussion, the task for the students was to make similar pyramids as an alternative for solving a series of textbook tasks. They could use addition or multiplication and had to design an easy and a difficult pyramid problem. The students did so with remarkable results (see Figure 7). Some students were careful in their attempts and created pyramids that were
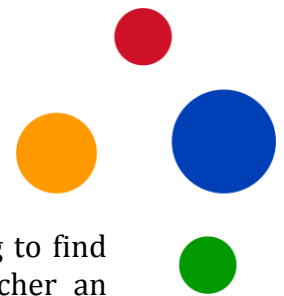
rather close to the textbook example, while other students were trying to find possible extremes. These productions by the students gave the teacher an overview of the algebraic expressions that were within the reach of her class, from very simple to very complicated including, for instance, fractions, decimals, and negative numbers.
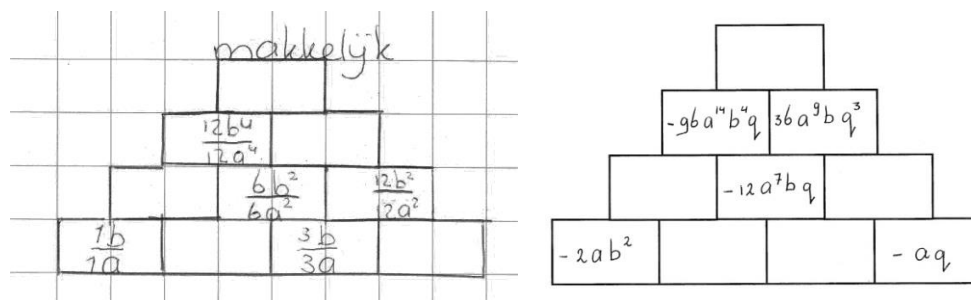

Figure 7. An easy and a difficult multiplication pyramid designed by students

During the activity, one pair of students posed the problem of the minimal amount of information that must be supplied before a pyramid can be solved. Other groups then took up this question and motivated them to design even more complex pyramids. The teacher reported that she noticed that her students improved their understanding of algebraic skills in this playful setting. Especially the moment where the students formulated the question about the amount of information needed was revealing, they tried many cases showing the scope of their algebraic skills and were practicing at the same time.

> *The students became owner of the mathematics, were motivated to do mathematics, and I could better see their capacities.* (Teacher from the Netherlands)

Normally students practice algebra with straightforward tasks, like 'expand' or 'factorize', and simply extend patterns without thinking deeply. With such tasks it is much more difficult to see what problems students encounter and whether they are able to use their algebraic skills in new situations. The teacher highly appreciated this change within the classroom but indicated that this was, respectively still is, a challenging process. Becoming familiar with these new teaching strategies needed for IBMT appears to be a process that requires time and attention.

Challenges when implementing IBMT

Teachers confront a number of conditions limiting the implementation of IBMT in day-to-day classroom practices. Main hindering factors are the mathematics textbooks that need to be covered, the time available to plan and implement IBMT activities, the available resources, and the assessment of students' work.

> *I think the biggest problem is the [class] time and time for planning of it. […] if you're looking at a very, very heavy amount of content in the syllabus then, fitting in the time to do inquiry-based learning is quite hard. Because you have so much to cover in a very short period of time.* (Teacher from the UK)
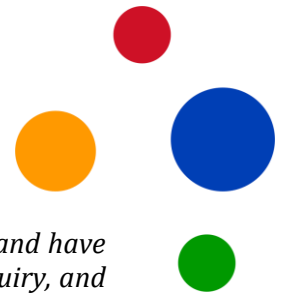
Lesson design is demanding. I have to take into account many variables, and have everything well planned, if I want my students to actively engage into inquiry, and to actually deliver a student-centered lesson. (Teacher from Cyprus)

Of course it takes a lot of time but it is not something additional. Actually, I learned to use inquiry-based learning to work on mathematical content. Students learn things in a much deeper way and understand more. (Teacher from Spain)

Another concern for teachers is assessing student performance. A main priority for teachers is in helping students to do well in their assessments. Examinations in schools mainly focus on students' capacity for reproduction skills. Therefore, teachers are in conflict to prepare their students for the exams or to implement IBMT within class.

My primary task is to prepare students for the next external assessment, which gives them a certificate that helps them in their future. They don't want more - and if I did more, well the first thing they would do is rebel. The next step would be that the parents would tell me that it is not my task to do this. (Teacher from Germany)

It is true that in many countries, examinations and tests do not directly reward students for their ability to inquire and solve non-trivial problem. This is an issue that some governments are aware of and are trying to address. A further potential hindrance for the implementation of inquiry-based learning concern students' behaviour, and was mentioned by some teachers in the beginning of their participation in PRIMAS. They initially feared that working with IBL within a class of 30 students could be problematic in terms of noise and disorder:

I thought "it's impossible to do that in my classroom because my students will not be thinking about the activity, they will waste their time, they will talk about something else and the noise will be tremendous". Then, I implemented that activity, and, I was surprised that everyone was involved and engaged, even as they were working in groups trying to obtain an answer. (Teacher from Spain)

Supporting factors for implementing IBMT

Scaffolding inquiry of students with well-planned questions (and other directions) is an important task for the IBMT teacher. By scaffolding we mean the use of teaching aids that have characteristics such as 'responsiveness' and 'fading'. Here, responsiveness means that the scaffolding is adapted to students' needs, and fading means that the scaffolding gradually disappears, as the students advance with their inquiry. The level of scaffolding needs to be adjusted to the level of the students. The teacher can vary to meet the needs of low-achieving students or to challenge high-achieving students. Within the lessons observed, teachers asked questions like: 'How can you simplify this problem? What assumptions might be made?' Then, after the students had formulated the problem, some teachers continued to ask: 'Can you think of a systematic approach? What is a sensible way to record your data?' As data was collected, others asked their students, 'Can you see any patterns here? Can you explain why these arise?' Towards the end, the teachers' focus was on communicating the findings: 'How can you explain this clearly and succinctly?' Asking these questions and sharing answers with the whole class supports the inquiry process.
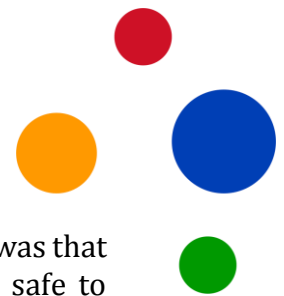
Another important aspect that became apparent during the observations was that teachers need to create a classroom environment where students feel safe to speak out and to make mistakes. Students not only need to feel safe in asking questions, making mistakes and stating their opinion, they also need clear signals about which behaviors are acceptable.

> *For example, I make mistakes too and students call my attention to it and calculate the task properly. I praise them for doing so and they like it. I think that is something that supports communication, too. Handling it this way: "Oh yes, I did it wrong. Sorry. You're right." I show students that mistakes are nothing bad. So they have the courage to show mistakes and to admit them.* (Teacher from Germany)

In retrospect, the main learning aspects of an inquiry-based approach are linked to fostering students' agency by providing clues, responsibility and confidence for performing an investigation. However, most teachers interviewed were particularly impressed with how this pedagogy motivated their students during mathematics and science lessons. Teachers talk about how inquiry-based learning links learning with fun:

> *In my opinion the students look forward to the IBL tasks as they find them fun to do and are different from a normal traditional lesson. Through IBL students are given the opportunity to discover, present their findings and have their say during a mathematical lesson whilst before the teacher was doing everything in class.* (Teacher from Malta)

Teachers emphasized that it is important to explain to students the new expectations that they have of them: that they should learn to actively ask questions, seek answers, compare approaches and pursue their own lines of inquiry – without continually asking for help. They should also know how important it is to learn to work collaboratively, just as professional scientists and mathematicians do in the world around them.

Conclusions

This chapter described ways to pursue IBMT, and classroom experiences with IBMT as reported by teachers. The experiences show that teachers encounter challenges when trying to implement IBMT in their daily practice. The challenges highlight the need for a shared sense of values, beliefs and aims of mathematics education. Mathematics education is not only aimed at supporting students' learning of algorithms and procedural fluency, but also needs to address competencies like creativity, dealing with missing information, making connections, critical thinking, collaborating and communicating. Tasks and teaching strategies that are inspired by processes of inquiry or that offer opportunities for inquiry-based approaches help to develop these competences. Furthermore, the teachers stress the importance of supporting conditions on the school level. Implementing IBMT asks for extra time to prepare and perform lessons and this needs support from colleagues and school authorities.

The challenges that teachers and students are confronted with while trying to engage in IBMT, cannot be overcome when the interventions are isolated and incidental. In order to be able to change a teaching and learning culture that
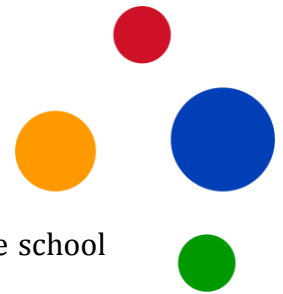
supports IBMT, it is important that the interventions also align with the school context and contribute to the demands of the curriculum.

A few important points can be made from the previous paragraphs. A successful implementation of IBMT asks for:
1. The availability of IBMT resources, not as isolated tasks, but as modules that show how mathematical topics from the curriculum can be approached in a IBMT-rich way;
2. Alignment with institutional conditions and constraints, including facilities and time available, and the official requirements and assessments of students and teachers;
3. A teacher learning community (at least one other teacher and preferably an experienced facilitator) for carrying out classroom experiments, discussing experiences and otherwise promoting teachers' professional development; opportunities to share professional knowledge with a broader (e.g. national) community, are also highly desirable.

MERIA design of modules rely on two frameworks that are related to IBMT, namely RME and TDS. These are introduced in the following chapters. It is important to emphasize that both frameworks have emerged from decades of research, and it is neither possible nor necessary to present all of their features, in order to enable readers to use and construct modules on their own. More readings are suggested in the references for those who wish to deepen their knowledge of one or both of the frameworks.
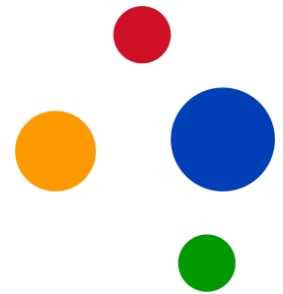
3. The Theory of Didactical Situations

Introduction

Let us consider an example of how to organize a lesson which enables students to engage in inquiry and to construct knowledge autonomously. In the lesson proposed, students discover a special case of the important mathematical result that similar figures (informally, figures of the same "shape") have proportional corresponding sides (i.e., there is a fixed "scaling factor" which allows to compute the sides in the second figure, if you know those of the first). This is sometimes referred to as Thales' Theorem. In fact, it can be generalised beyond polygonal figures, but this is far beyond the secondary curriculum – even if similar, non-polygonal figures appear in many real-life contexts, such as pictures shown in different sizes. Notice that the difficult notion of angle is not needed, and may not appear at all, in this activity.

The students get the following instruction:

Here are some puzzles (Example: "tangram", Figure 8). You are going to make some similar ones, larger than the models, according to the following rule: the segment that measures 4 cm on the model will measure 7 cm on your reproduction. I shall give a puzzle to each group of four or five students, but every student will do at least one piece or a group of two will do two. When you have finished, you must be able to reconstruct figures that are exactly the same as the model (Brousseau, 1997, p. 177).
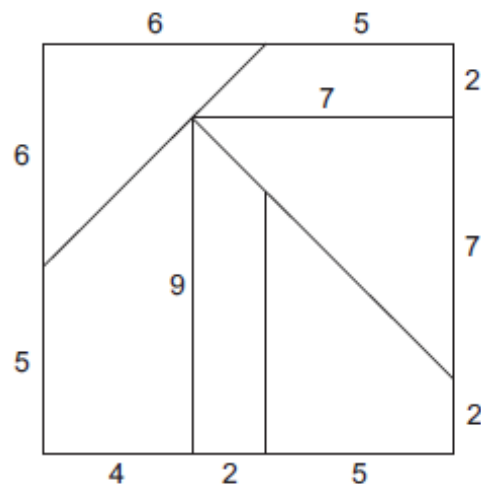

Figure 8: The puzzle used in the puzzle situation

After receiving the problem, students will start working without help from the teacher. The problem is posed to the students who are used to the idea of increasing the amounts by adding. However, when students add 3 cm to each side they cannot reconstruct the larger puzzle by assembling the pieces, as they do not fit. Students try to employ previously developed knowledge (enlarging by adding) to solve a problem, but the puzzle situation forces them to realize that the
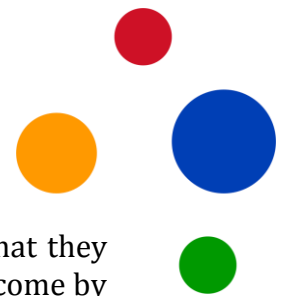
reasoning they have used in the past is not working in this case, and that they need to develop new mathematical knowledge. The obstacle can be overcome by changing the strategy to a multiplicative method to increase the side lengths. It is important that students develop the ideas of proportionality by themselves, and in particular come to see the multiplicative model as required by the situation, not simply by the teacher.

In the next step, the teacher invites all groups to formulate and present their findings. For example, some of the students might be invited to present and explain what they did to enlarge their piece of the puzzle. Also, groups report on whether their enlarged pieces fit together or not, and what they plan to do.

To construct the new puzzle correctly, each student from a group must come to the same hypothesis that the side lengths of all of the pieces must be multiplied by the factor 7/4. The teacher is sure that the students have reached the desired goal of understanding proportionality if they validate their strategy by assembling the new puzzle.

In the end, the teacher may formulate the idea of proportionality between geometric shapes in a formal way. Based on the discussion during the lesson and final reflection on the problem, students' personal ideas become shared knowledge similar to what can be found in different media such as textbooks or online resources.

The design of this situation of teaching is a classic product of the Theory of Didactical Situations. In the rest of the chapter we will present basic notions and principles of the theory, illustrated with further examples and guidelines on how to use this approach in the classroom.

Personal and institutional knowledge

The Theory of Didactical Situations (TDS), which was initiated by Guy Brousseau from the late 1960's, has produced several ideas and results which can help teachers deploy and develop their mathematical knowledge, as they work with the design and orchestration of teaching. TDS supports teaching that lets students be the inquirer and constructor of mathematical knowledge in a way that captures the essential features of IBMT.

In TDS, it is very important to distinguish between two kinds of knowledge:

Institutional knowledge (sometimes called *public* or *official knowledge*) is the knowledge presented in textbooks, webpages, research journals and other shared resources. It represents a synthesis of mathematical activities, done by individuals but subsequently validated by others, and made public. In these resources, mathematical knowledge is presented in a concise and precise form, while the inquiry process which leads to its development is usually not visible. This deductive way of presenting mathematical knowledge is shared and evaluated by a community of researchers, teachers and occasionally by the public
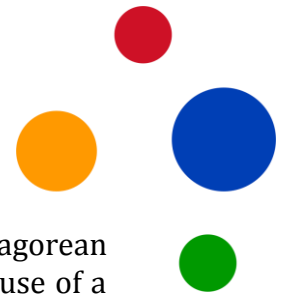
in general. A simple example is given by the presentation of the Pythagorean Theorem as $a^2 + b^2 = c^2$, where a and b are the legs and c the hypotenuse of a right angled triangle. Today this formula is the "institutional knowledge" that teachers introduce to their students and which is remembered later on in life, rather than a geometrical idea or argument behind it. Institutional knowledge is also sometimes called shared, public or official knowledge.

Personal knowledge is the knowledge that students (and others) construct while interacting with a mathematical problem. These ideas or knowledge will often be implicitly given and related to the context they are developed in. Student may develop personal knowledge about the Pythagorean Theorem by playing with triangular and square tiles, as illustrated in Figure 9. It clearly takes more to establish the official form described above.
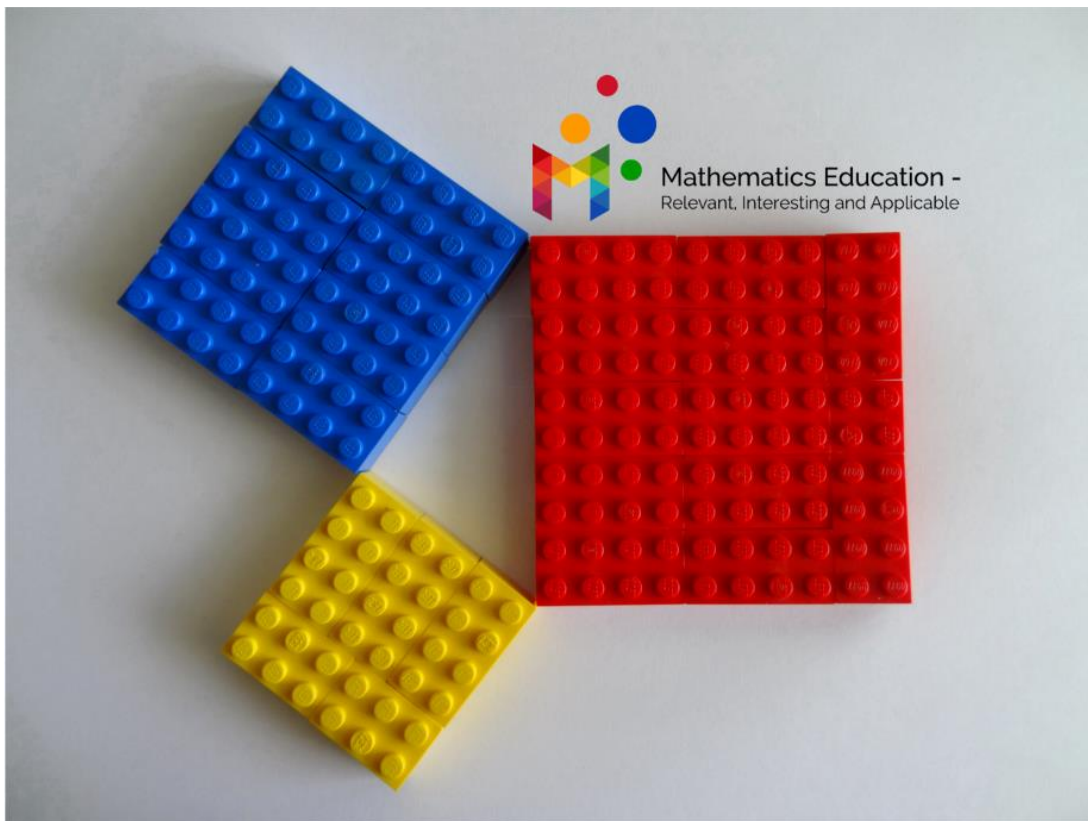

Figure 9. Tiles assembled to illustrate Pythagorean Theorem

In the puzzle example, the personal knowledge which students construct at first, concerns only the success or failure of specific methods to magnify pieces. The official knowledge aimed at is that if to figures *A* and *B* are similar (have the same "shape"), then the ratio between corresponding sides (*a/b*, where *a* is a side in *A*, corresponding to the side *b* in *B*) is constant. Several situations may be needed to reach the official knowledge in something close to this level of generality. At the end of the puzzle situation, one may at most reach the consensus that only multiplication by 7/4 seems to work in the given case.
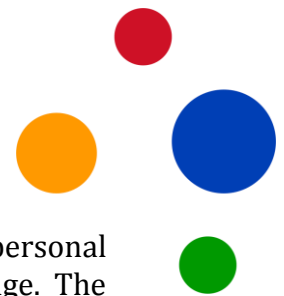
When students interact with a mathematical problem and develop a personal answer to the question at stake, they expand their personal knowledge. The students' personal knowledge is most likely to be slightly different from the institutional knowledge. It will be further developed and formalized when shared and discussed with others. Hence the communication with classmates or peers will further develop and formalize the students' initial ideas.

It is essential that the teacher challenges his students' personal knowledge by posing new problems which require knowledge they have not yet fully developed. In this way, personal knowledge is being validated. It can be validated either by the teacher, by the problem situation itself, or compared to other students, e.g. to their strategies when solving a problem. In this way, personal knowledge is transformed and becomes more formalized. This means that the knowledge becomes closer to what can be regarded as institutional knowledge.

Didactical and adidactical situations
The distinction between personal and institutional knowledge presented in TDS enables teachers to organize lessons using inquiry situations, i.e. IBMT approach. A part of the idea in IBMT is that teaching should offer students the opportunity to engage in activities similar to those of a researcher.

A key component of designing such situations is the notion of the *didactical milieu*. The milieu is the environment with which the student interacts to obtain new knowledge. It consists of the problem, students' previous knowledge, and artefacts such as pen and paper, ruler, calculator, CAS-tool (Computer Algebra Systems), a puzzle etc. When preparing the lesson, the teacher specifies the *target knowledge* and designs an appropriate milieu for the students' development of that knowledge. However, milieus can be more or less appropriate with respect to developing a certain piece of knowledge. In the puzzle situation described above the milieu consists of the puzzle, new sheets of paper, scissors, rulers, and students' previously developed knowledge. The (epistemological) obstacle which the students encounter stems from the mathematical nature of the problem. Hence the milieu carries a high potential for the students to construct the intended knowledge without the teachers lecture neither on the proportionality between geometric shapes nor the principles of similar triangles. The milieu creates among the students the need to construct this knowledge.

It might happen that not all of the students consider the implications of the multiplicative strategy on preservation of the angles, though this is needed if the pieces should be joined together in a new large square similar to the initial one. Hence the correct strategy leads to the development of the intended institutional knowledge. The TDS approach to teaching and learning is often described in terms of a game. The design of the situation and its milieu can be compared to outlining a field for a sports game, and formulating the rules of that game. When the students win the game, they have developed the optimal strategy for the game. Hence, winning corresponds to learning and the optimal strategy means that the students have developed the intended knowledge and methods. In other words,
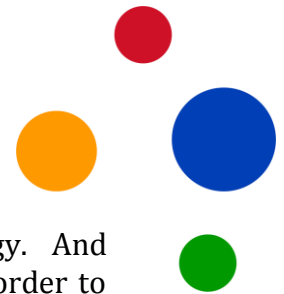
the game creates the need for the development of the winning strategy. And designing the "field to be played in" (the situation) should be done in order to maximize the potential for students to find this strategy.

When the milieu is properly designed, the students can interact with it autonomously, without further guidance from the teacher. *Adidactical situations* are those where the students are engaged in the problem and explore the milieu without the teacher's interference. In these situations, the students are developing their personal knowledge, by adapting it to the problem they work on, through further inquiry activities and testing of ideas in the milieu, or through formulation of arguments, as they try to convince peers.

Didactical situations are those where the teacher explicitly interacts with the students, in order to further their learning of something specific. Indeed, *didactical* refers to the *intentional act of someone to share some knowledge with somebody else.* One main function in didactical situations is to initiate, regulate and moderate adidactical situations, and to ensure that the knowledge developed there becomes shared, validated and (for the relevant parts) recognized as "correct". As shown in Figure 10, means that didactical situations consist in the teachers' interaction with adidactical situations. The adidactical situations could, of course, be more or less rich in potential – from tacit listening to teachers' explanations, to active engagement in a rich problem situation. Indeed, the main learning potential of students lies in adidactical situations since those carry the potential of students developing their personal knowledge, which can become shared knowledge through the didactical situations. In other words, the learning potential lies in the dialectic between adidactical and didactical situations or between personal and shared knowledge. Figure 10 also shows how didactical situations as a whole consist in a "double game": the students' game with the milieu (adidactical situations) and the teachers' game with the adidactical situations (which she plans, devolves and regulates).The figure, in particular, shows that an adidactical situations does not imply the teacher is absent or inactive. Spontaneous self-study is not an adidactical situation; it is non-didactic.

Adidacticity is a special phenomenon *within* didactical situations: the person who wants to share some knowledge can *purposefully* withdraw from the interaction, in order to let the learner act in ways which are useful or even necessary to obtain the knowledge. This is a very general phenomenon, which is not at all an invention by TDS; one observes some element of adidacticity in most didactical situation. Of course, the quality of the students' autonomous acts depends on the milieu.
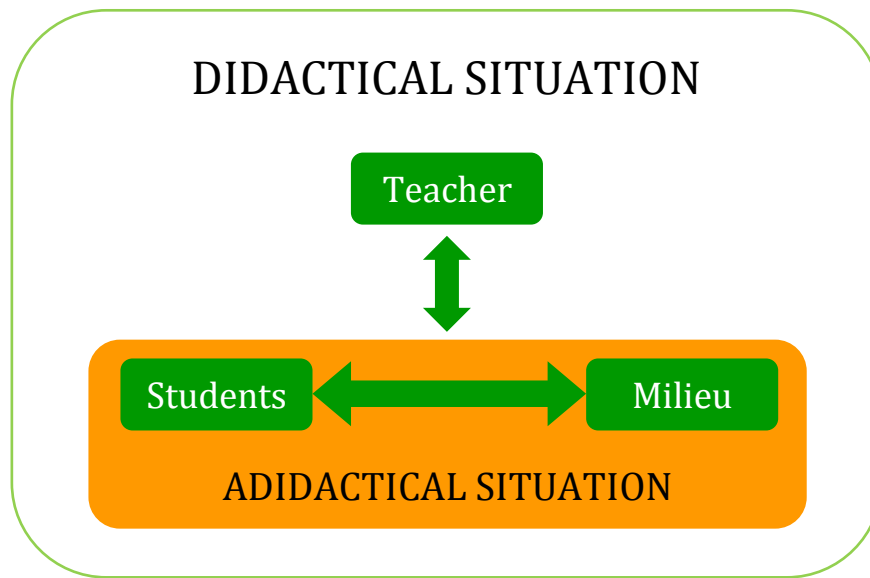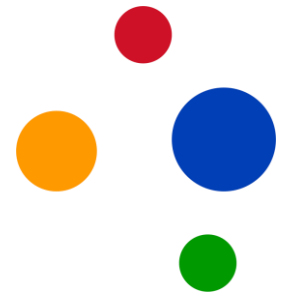
Figure 10: Didactical situations as a double interplay

The role of the teacher

It is important to clarify how the teachers' role in a TDS based design is different from what many teachers may be used to in their normal teaching.

In much commonly observed mathematics teaching, the teacher first introduces a new notion, method or theorem. Then he shows examples in which he uses the new knowledge, after which the students imitate the teacher by solving similar exercises. Finally, the students' work is assessed by the teacher. In such an approach, the teacher starts by institutionalizing the institutional knowledge. There is no milieu for the students to explore – or at least it is a poor one with no room for the students' inquiry processes, since it is evident that the winning strategy is to imitate the teacher's examples. While solving the exercises, the students are active and probably formulate answers in notebooks, but this is not inquiry, as they know the good method (assuming they have listened in the first place). The validation relies entirely on the teacher who approves or rejects the students' answers. In this setting, the teacher is the source of all true knowledge; the students merely absorb and follow the good example of the teacher.

The approach has drawbacks. When the Institutionalisation is placed at the beginning of the teaching, providing the students with all the relevant official knowledge and merely asking them to "apply" it in specific cases, the students may not construct appropriate personal knowledge – in some cases, they merely adopt the official knowledge as a tactic to solve certain tasks given by the teacher or posed at exams. They may keep, in parallel, contradictory ideas and beliefs, including misconceptions. This ruins the potential of a rational process of knowledge construction, which is placed in the dialectic between personal and institutional knowledge, as illustrated in Figure 11. When students think, talk and write about the exercises, they try to do what their teacher expects and rewards, even when they have no clue as to what it means or why it works. The learning

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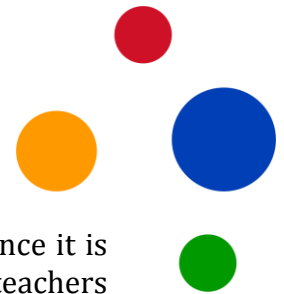
outcome of repeating the solution strategy may be surprisingly poor, since it is fully depending on surface recognition of the tasks it applies to. Most teachers have witnessed absurd cases of how students fail in such an approach.

By contrast, when the public or institutional knowledge is clearly recognized as consistent with knowledge the students themselves have constructed, while interacting with a suitable set of problems (in adidactical situations), the public knowledge appears to them as rational and well-founded, and they will more easily transfer it to new types of problems, as they know the rationales behind it. But to achieve this better result, the teacher will need to abstain from merely telling and training. This requires much more of the mathematics teacher than is normally assumed.

In fact, according to TDS, the role of the teacher must design or select situations in which students can develop personal knowledge corresponding to institutional knowledge, *including* the rationales of the latter. Also, the teacher needs to orchestrate the dialectics shown in Figure 11, which is cyclic: to teach new knowledge, the teacher designs and devolves a mathematical situation, in which students may develop their personal knowledge. The teacher also needs to help students share this knowledge in the public domain of the classroom, where it can eventually be aligned with the new knowledge which the students should acquire. What *teachers* should know is not just, or primarily, the institutional knowledge; it is the *situations that enable students to acquire the knowledge*.
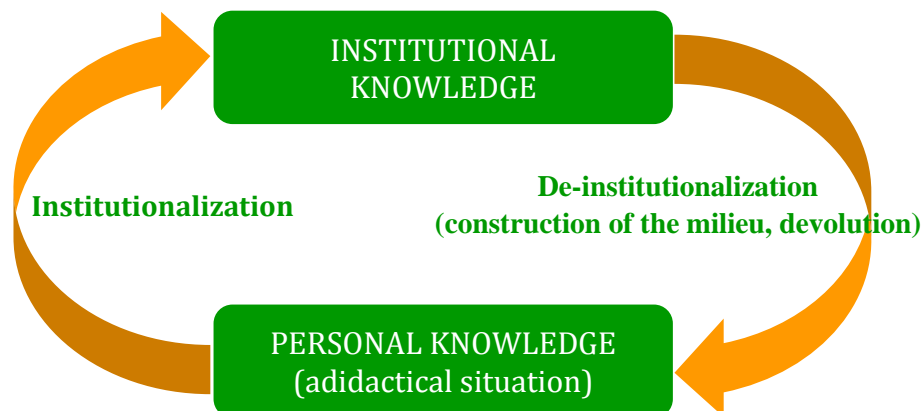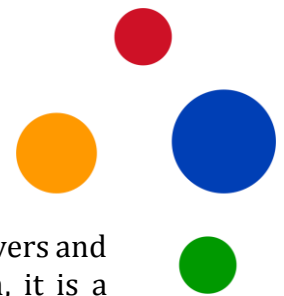

Figure 11: Interplay of personal and institutional knowledge in didactical situations

Didactical contracts

Even for experienced teachers it can be a challenge to navigate in inquiry based scenarios, as they should not simply feed the students with official knowledge and oversee their digestion of it, but instead, they must guide students in their own

construction of the knowledge. When the teacher knows all the right answers and sees students take wrong or less favourable approaches to a problem, it is a genuine challenge not to correct or guide the students in the direction of the best strategy. It is very important that the teacher knows what parts of the knowledge have to be constructed by the students in adidactical situations (without teacher's intervention) and which may be institutionalised directly by the teacher. In a teaching situation, students and teachers have mutual expectations about each other's roles and responsibilities in the classroom. The collection of these expectations is called the *didactical contract* of the situation. It is not a contract in the ordinary sense of a written document; still, we can observe its effect in teachers' and students' actions. For instance, when students asks the teacher to tell them if their solution to a linear equation is correct, they show their expectation that it is not (yet) their responsibility to control the correctness of such a calculation. The teacher may act accordingly and tell them his opinion, or he may try to adjust this part of the contract by organising a student game with the new problem (what techniques exist to ascertain the correctness of a proposed solution to a linear equation).

If students are used to a teacher providing them with answers from the very beginning, a certain amount of frustration can occur when they are given open ended inquiry based activities. In these situations, students will often ask the teachers, more or less indirectly, to provide them with the expected strategy. It can be tempting for the teacher to explain the students what to do – it's clearly easier for everyone. However as explained earlier, this will ruin the learning potential of the teaching situation. To prevent such frustrations it can be helpful to begin by explaining to the students that the teaching is going to change and that they are expected to engage in solving problems even if they feel unprepared.

When the students start to experience that the teacher's Institutionalisation in the end is merely a reformulation of personal knowledge they have constructed themselves, they will feel that what they were doing is meaningful and important, and will gradually accept their new roles and responsibilities. There is lots of evidence that students will also, over time, develop a more positive relationship with mathematics as a whole: instead of a meaningless and endless inventory of given answers, mathematics will appear to them as a rational, challenging and satisfying enterprise – much as it is to successful researchers.

The phases of didactical situations

The idea of TDS is to create situations that address a well-known obstacle regarding a piece of mathematical knowledge, which create the need for the students to develop or construct new mathematical knowledge. To design and calibrate such situations is a core element of TDS and its "didactical engineering".

Teaching situations are organized into five phases. We will describe each of them together with comments on their role regarding students' learning. The sequencing of the phases is not strictly given and an overview will be provided
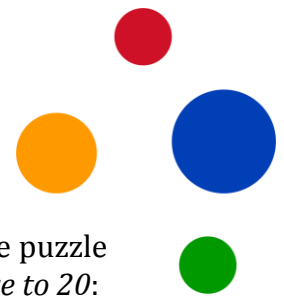
later. We will illustrate all phases using two examples. The first one is the puzzle situation presented in the introduction, and the second is the famous *Race to 20*:

The students are supposed to play a game where the winner is the one who first reaches the number 20. Two players play against each other. One player starts the game by choosing the number 1 or 2. The other player adds 1 or 2 to the previous result both striving at being the one to say 20.

The goal of the Race to 20 is to learn how division provides the answer to a new class of problems, along with providing students with an early acquaintance with proofs (to justify "winning strategies"). Concretely, students are expected to find and justify that all the winning numbers are 20, 17, 14, … , 2 (numbers less than 20 whose remainder after division by 3 is 2). Experimentations with the situation confirm that students discover these numbers as partial winning strategies and in that order, and thus build up final strategy (rest by division by 3) gradually, and on the way in fact also discover some basic objects and principles of modular arithmetic, such as (what to the mathematician is) a specific congruence class.

Devolution phase

The first phase is called the devolution phase. In general, devolution is a transfer or descent of something to a lower level. In TDS devolution is the starting point. It is the phase in which the teacher presents the problem and explains the rules for solving it. In other words, the teacher hands over the milieu to the students. In the game terminology, the teacher presents the playing field and the rules of the game. It is important to make sure that the students have understood the rules and are able to engage in the intended activities when the devolution phase is completed. In the puzzle situation, it is obvious how to play the game in the sense that students should create new pieces and construct the enlarged puzzle. In this phase the teacher does not provide more help. In the devolution phase of the Race to 20 the teacher presents the rules but also engages in one game with a student, which serves as a demonstration at the blackboard of how the game is played. Whether the teacher decides to hand over the milieu through an example of the activity or simply by presenting the milieu with its rules and artefacts depends on the concrete problem and situation.

Action phase

In the action phase students autonomously engage in the problem. In the puzzle situation, students will initially employ their previous experience from mathematical problems of increasing magnitudes by adding 3 cm to all sides of the geometric shape they were given. To employ previously developed knowledge and experience is a natural initial hypothesis, even if it proves to be wrong.

In the example of the Race to 20, the students are asked to play the game together with the person next to them. In the beginning the students' work might be based on "trial and error" and not carry any explicit strategy. Though, experience with the game might gradually indicate that the person who says 17 can win the game no matter what the other player adds to 17.
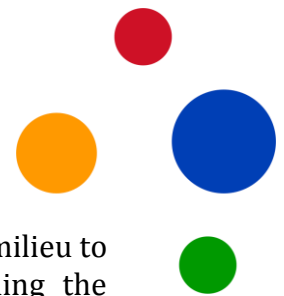
What the two examples have in common is that they both contain a rich milieu to support students' development of some personal knowledge regarding the problem they are engaged in solving. In this phase, the knowledge may be rather implicit and rudimentary, and it can be difficult (if possible at all) for the students to formulate the assumptions involved in the action they take. The students draw on their previously developed heuristic competences, but at the same time, develop them further. The phase can also be said to have similarities with researchers' first approach to an open problem. They might know the regulation of their game or their milieu as the definitions, lemmas, and theorems of their research field, as well as the generally accepted mathematical techniques associated with this field. But they still may just play with assumptions, and have no definite "theorem" to prove.

Formulation phase

In the formulation phase students are required to present what they did in the action phase; initial ideas, hypothesis or simply what they have tried to do so far. This can be arranged in different ways in the classroom, but it is not always enough to require the students to engage in a classroom discussion. First of all it is often the same eager few students who engage in classroom discussions. This is a problem if we want all students to engage in inquiry based learning. In IBMT communication and personal hypotheses must be shared and commented on by peers in order to formalize the personal knowledge of each student, which starts to take form in the students' minds while dealing with the problem in the given milieu. This means that all students need to formulate their personal ideas and should present as well in the formulation phase. Often this can be done in minor groups.

In the puzzle situation, the gathering of pieces represents a formulation phase where each student presents and explains his or her strategy of constructing a new piece of the puzzle. The students are, as a group, expected to encounter the obstacle of pieces not fitting together. Hence, it will lead students to discuss the strategies they have already tried out and new ideas regarding other approaches could possibly be developed. Even if one student got the right strategy from the beginning, this person must convince the rest of the group through mathematical arguments, which can be understood and accepted by the rest of the group.

In the Race to 20 example, the formulation phase is carried out as a new round of games where each neighbour team of two players play against another team. To agree upon a shared strategy forces the students to share personal knowledge and convince each other upon what can be the optimal strategy of their team. Again, this verbalizing of experience and ideas is an initial step towards the creation of institutional knowledge. The functioning of the formulation phase is to create a situation where students are forced by the milieu or the regulations of the game to articulate the experience and ideas they gained from acting upon the problem initiating the construction of elements of mathematical theory.
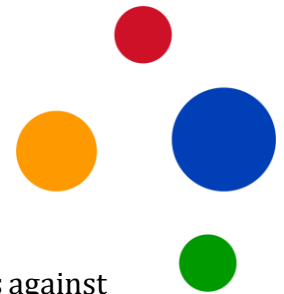
Validation phase

In the validation phase, students are testing their strategies or hypotheses against the milieu. This means that the work of the students can be validated without the teacher telling them whether they are right or wrong. The mathematical problem will to some extent provide the students with the answers regarding the viability of their answer or strategy if the situation and milieu is designed strong enough to do so.

In the example with the puzzle the milieu is designed in a way where the pieces of the puzzle cannot be put together and form an enlargement of the first puzzle if the students use an additive method, or in fact *any other method* than the multiplicative one. Therefore, when students initially might choose the less productive strategy the validation phase will show them that their idea was wrong and that they need another strategy, which later can be validated in the same way. It should be stressed that a productive strategy should be within reach of the students' mathematical ability. If the students get "stuck" in a problem with no ideas on how to proceed, this of course will be counter-productive with respect to development of mathematical curiosity and inquiring mind.

In the Race to 20, the validation is largely a matter of having the strategy that wins every time. The winner is assumed to have the strongest solution strategy. If not, both teams could develop the optimal strategy. Or both teams continue with no real strategy. Even if the winner has the best strategy of the two teams, the team might not have developed the optimal strategy from the very beginning of the game. In such a case the teacher can choose to divide the class into two large groups, ask them to prepare optimal strategies and in the end battle against each other. This can be regarded as an additional formulation phase where students try to convince each other about strategies. Finally, the last match can take place where the winner's strategy will prove to be the strongest – be validated so to speak.

Institutionalisation phase

The last phase is the institutionalization and here the personal knowledge will finally reach the state of institutional knowledge. This phase will most often be carried out by the teacher gathering ideas, summing up main points of the shared strategies and present it as one optimal strategy. The presentation of what is being institutionalized will often be a presentation of mathematical knowledge being concise and accurate as in the textbooks.

In the puzzle situation, the teacher might introduce the informal idea of similar figures, as "one being a magnification of the other" (no magnification being "same shape and size"), using a discourse similar to official guidelines or textbook material for teaching at the given level. The fact that sides are proportional is conveniently expressed through the multiplication by a common factor (7/4) which was shown, experimentally, to produce a magnified puzzle, similar to the original. More advanced ways of articulating the relation between the initial and the created figures in terms of mapping could be too advanced for 14-years old.
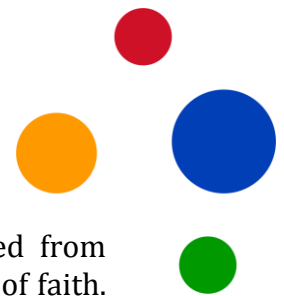
The essential point is that the mathematical knowledge is established from experience and reasoning, rather than being merely posed as an article of faith. This is in a sense even more difficult to achieve at a level where more formal definitions and arguments are still out of reach for the students.

In the example with the Race to 20, what is being institutionalized is the winning strategy. This can be reduced to the list of numbers a player should strive to say in order to control the game and win it in the end. It might be relatively easy for students to point out the number 17 as important as mentioned earlier. The winning numbers or the strategy can be written as the numbers $N = 3n+2$, where n is the number of a given round in which w is the winning number. If you start by choosing number 2 you can have control all the way through. For the students to gain that insight might require that the teacher continuously encourages the students to improve their winning strategy. Depending on the level, one may eventually analyse the more abstract game "Race to N while adding one of the numbers 1, 2, … , n at each step. And then modular arithmetic is around the corner.

The amount of mathematical detail presented by the teachers in this phase should align with the activities carried out by the students. It should be a synthesis of the knowledge constructed by the students in order for them to recognize and relate their personal knowledge to the knowledge being institutionalized and deemed the shared knowledge of the class.

It is important that this phase does not end as a lecture making the students actions needless – the institutionalisation should be a continuation of the shaping of students' knowledge regarding the given problem. If the teacher starts lecturing and goes beyond the students work, the teacher risks that the students perceive their actions as an excuse for the teacher to lecture on the topics which really matter. In those cases, the students are not likely to treasure or engage in mathematical inquiry and autonomous construction of knowledge, but will imitate the teacher when doing mathematics.

On the importance of adidactical situations

In teaching situations where the students are not progressing as expected, the teacher may feel tempted to move to phases where they are in control of the situation. However as hinted, this usually ruins some of the potentials for students' learning. The devolution and Institutionalisation are didactical situations. The action is an adidactical situation, and the last two can be somewhere in between; but one should strive at maximizing the role of adidactical components. In particular, elements of adidactical validation – without the teacher as arbiter – are often crucial to ensure that students develop a fully rational relationship to the target knowledge, rather than a hit-and-miss approach where the "hit" is only recognised through the teachers' approval. In general, we speak of the *adidactical potential* of a didactical situation – that is, the potential for students to work autonomously with its mathematical problem, and based on that, reach the target knowledge It is an important idea for teachers to seek to realize the full adidactical potential of the situation – through appropriate
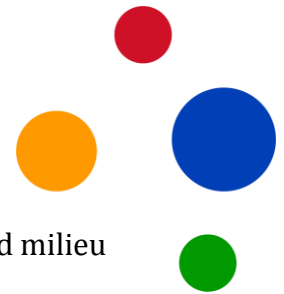
choices in the phase of devolution, and by carefully adjusting the devolved milieu to the students' capacities (obviously, without trivializing the problem).

Summary of the phases

In Figure 12, we provide an overview of the five phases that make up didactical situations.

	Role of teacher	Role of students	Milieu	Situation
Devolution	Introduces, hands over the milieu	Receive, try to take on a problem	Is being established	Didactical
Action	Observes and reflects	Act and reflect	Problem is being explored	Adidactical
Formulation	Organizes, if needed initiates through questions	Formulate as specifically as possible	Open discussion	Adidactical or didactical
Validation	Listens and evaluates if needed	Argue, try to follow others' arguments	Guided discussion	Often didactical
Institutiona-lisation	Presents and explains	Listen and reflect	Institutionalised knowledge	Didactical

Figure 12: An overview of the TDS phases, their functioning and actions of participants in the teaching and learning (translated from Winsløw, 2006, p. 140)

As stated in the beginning, the five phases are not just used as design tools, which only apply to teaching developed on the basis of TDS. The phases can be used to analyse any mathematics teaching (for instance to identify if some phases are missing or underdeveloped). Even if the teaching is very different from what we have presented in this chapter, the phases still apply to the analysis of the teaching – and for teachers, they provide an important instrument to distinguish crucially different parts of their teaching, which have distinct roles and effects for their students' learning.

The dynamic use of the phases

In the two simple examples used throughout the presentation of the phases, it is evident that each of them are designed in ways so that the didactical milieu sustains the students in their actions, lets them experiment and formulate hypotheses (both good and bad), and provides conditions that are strong enough to validate these hypotheses. Concretely, the puzzle pieces are impossible to assemble, or the students keep losing some games. The two situations also represent somewhat strict interpretations of the phases and how they are linked. What happens if the teacher hands over a milieu with a problem which the students are unable to solve?
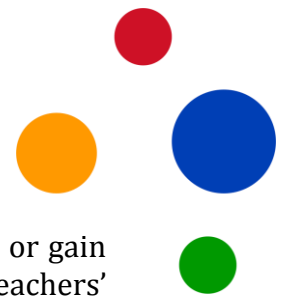
When designing mathematics teaching, it is of course important to have or gain some insight regarding the knowledge students already possess. The teachers' knowledge of this can be based on the curriculum for middle school mathematics, the textbook which the class has used previously, or other resources indicating expected outcomes. But even when students are supposed to have learned something it can be a good idea to "check" what they actually remember from earlier classes, as part of the devolution phase.

A straightforward way of checking this is to ask the students e.g.: do you remember the Pythagorean Theorem? Although this carries the risk that students are unwilling or afraid to admit that they cannot remember the theorem. Some students will do this to please the teacher. Others are afraid to "lose face". A more fruitful way to ask could be: what do you know about right-angled triangles? If they do not mention the expected knowledge the teacher might need to devolve a new problem to the students before handing over the intended problem and milieu. This new problem should provide the students with the possibility of rediscovering the knowledge they might have forgotten and get a shared starting point, by reconstructing the knowledge they were expected to possess already.

A similar problem can be encountered during the action phase: the students might misunderstand the devolution. In the puzzle example, they do not have a clue about alternatives to additive enlargement of side lengths of the shapes. How to overcome such a challenge in the teaching situation depends on how many students are unable to engage in the problem and the students' mathematical achievements in general. In these situations, the teacher must have thought through how to regulate the milieu. The risk here is to give away the target knowledge, which the students are supposed to construct. In the puzzle situation, the teacher can initiate a formulation phase where students share their preliminary ideas, and then make a table showing side lengths in the given puzzle in one row, and the magnified side lengths in the other. This may give rise to the idea that more "methods" may exist to magnify, as a rudimentary idea of functions. Indeed, 4 can become 7 as a result of more than one calculation. A crucial question which may arise and be discussed is: what happens to a side of length 1? Does it really become 1+3=4 after magnification? Considering that a side of length 4 is composed by four pieces of length 1, may provide a clue;, as four magnifications of 1 should form, together, a side of length 7. With such considerations, coming as much as possible from the students themselves, even the students who had no ideas in the beginning might be able to develop a different approach to the given problem. This means that the phases can be used dynamically, in a controlled way. Depending on the students' engagement with the problem and milieu it can be reasonable to move back or forward in the phases in order to secure that everybody can act and construct some personal knowledge regarding the problem at stake. The interplay between personal and shared knowledge is a crucial dynamics which can be controlled by systematic and planned use of the phases.
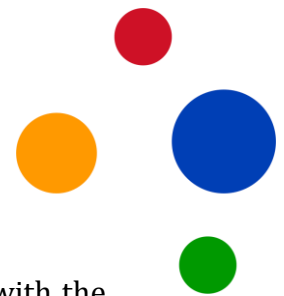
A more elaborate example for high school

In this section we present an example of a TDS based teaching design with the learning goal "to introduce the idea and some methods of optimization". The problem, which the students should engage in, is the following:

> You are given a string of length 1m. This string should be divided into two pieces. One piece is used to form a square and the other to form an equilateral triangle. The question is, where to cut the string in order for the two geometric shapes to cover the minimal area together?

One strategy leads to a precise answer, theoretically developed, but inconclusive. It still requires a considerable amount of algebraic operation based on students' knowledge of, for instance, geometry or regression. In both cases the need for new methods for solving optimization problems becomes visible.

The milieu consists of the problem, actual strings (e.g. 5 strings per group of students), a ruler, scissors and maybe a calculator or a computer. The devolution phase is initiated by the teacher, who asks the class: "what do you know about areas of geometric shapes?" This is assumed to remind the students about formulas such as the area of a square and triangles:

$$A_{square} = s^2 \qquad (s \text{ being the side length}),$$
$$A_{\text{triangle}} = \frac{1}{2}hb \qquad (h \text{ is the height and } b \text{ the base length})$$
$$A_{\text{triangle}} = \frac{1}{2}ab\sin C \qquad (\text{where } a \text{ and } b \text{ are side lengths in the triangle and } C$$
$$\text{is the angle between them}).$$

Other shapes might be covered as well. After sharing this institutional knowledge on areas, the students are divided into groups. Each group gets five strings, scissors, and a ruler; they are allowed to use calculators or computers if they feel like it. The groups are now handed over the problem. This entire phase is a didactical situation where the teacher is the moderator of the classroom dialogue.

In the action phase, students are starting to work on the problem. Here a number of strategies can be chosen and we will mention three of them. Some students might choose a "trial and error" strategy meaning: cutting a string, creating the two shapes, measure and calculate the area of the square and the triangle. The same string can be used to create two pairs of shapes. Then the next string is cut, followed by new measurements etc. In the end the students might get the sense of where the optimal cut will be based on their experiences. Other groups might get the idea of using such measurements as data. These can be depicted in a computer program, graphic calculator or drawn with a pen on paper producing a graphic representation of where to cut and the sum of areas. If data is well chosen in the sense they cover the entire string including the area of the minimum point, these will indicate a parabola. If the data points are plotted in a computer program the students can carry out a regression to get a formula of a function describing the data. Depending on the strategy and tools available to the students they can
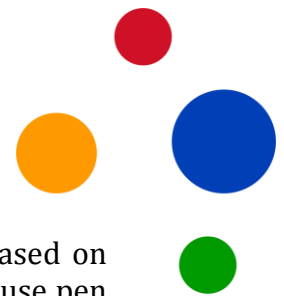
find the minimum of the area function, estimating the least y -value based on where the data point is placed in the coordinate system. If the students use pen and paper, they can draw an approximation of the parabola, which describes the data points. If the students use a graphic calculator or a computer program they can use regression to find the formula describing the relation between the area and the point where they cut the string. Using a CAS-tool the students can find the extremum point by letting the program analyse the graph or they can do it simply by looking at the graph. If the students have carried out the correct regression they will get a formula on the form

$$f(x) = ax^2 + bx + c,$$

where f is the area and x is the length of one piece of the string, and in this case the least total area is given by

$$y = -\frac{b^2 - 4ac}{4a},$$

which corresponds to

$$x = -\frac{b}{2a}.$$

Students choosing this latter strategy of course need to have been taught about (second degree) polynomials, parabolas and how to calculate the extremum points of those.

Other groups might consider the problem as an algebraic problem. If the 1m string equals 4 sides of the square, *4s*, and three sides of the triangle, *3t*, then we get the equation $1 = 4s + 3t$. Next, the students can express the total area as

$$A_{\text{total}} = A_{\text{square}} + A_{\text{triangle}}.$$

This function is a second degree polynomial where you can find its least value by the same methods as described above, if the regression strategy was chosen.

This phase is adidactical. The teacher refrains from interfering with the group work, but can assist with guidance on how to manage the CAS tool, calculator or other more practical problems, if needed. At the same time, the teacher gets an insight of which groups have chosen which strategy or what kind of challenges the groups are dealing with during their inquiry. This concrete example captures the idea of an open ended approach, where students are posed a problem, which might have a multitude of solution strategies all converging towards one answer.

After the first brief phase, students are asked to present their strategy for solving the problem. To verbalize their actions helps students to become explicit about their somewhat vague ideas and hypotheses from the inquiry process. One can argue that the group work represents the first formulation phase in the sense that students in each group need to agree upon strategies or hypotheses in order to work together. Further the action in groups might lead group members to reject ideas and pursue others. This process can also cover elements of validation. From the first experiences of cutting, measuring and calculating areas students might believe that now they know where to cut the string. But a third calculation might lead to an even larger area than the first two calculations. The group then needs to consider their strategy. Hence in the group work all of the adidactical or

potentially adidactical situations might have taken place before sharing strategies with the other groups.

When students have reached an initial answer to the problem, the groups are required to present their work to the rest of the class. As a first presentation, the groups could simply be asked to give the length of one of the pieces of the string in order to see if everybody agrees upon where to cut. If the groups do not agree there are even more reasons to present their own and listen to the other groups' strategies. It is expected that this will make some students realize that a "trial and error" strategy is less powerful when striving for a precise answer, but also that their work so far can be used if they change to a regression strategy.

Here the formulation phase can have an overlap with a validation phase. All the suggestions for where to cut the string can be tested in the milieu. For each suggested length, the total area of the square and triangle can be calculated. In this sense the milieu can assist the validation of which group has proposed the cut relating to the least area. The challenge here is to engage students in discussing their chosen strategy as well. Therefore the teacher can ask if we can be sure that there is no better choice of a cut. This means if the class has accepted the task to be solved by "trial and error", they now need to engage in more precise arguments as well.

A challenge for those students who choose regression is to find out which kind of function actually describes the situation in the best possible way. If they only have data from below or above the extremum point they might consider the relation to be e.g. linear or exponential. To avoid these situations the students must be asked to what extent these relations actually make sense. This can be regarded as a new devolution of a slightly different problem within a similar milieu.

The optimal strategy cannot be validated by testing against the available milieu. Therefore the teacher plays a more active role in this part of the validation, but it is still important that the rest of the class is also convinced about the presented strategy.

In the Institutionalisation phase it is important that the teacher sums up the ideas and relates them to each other. For example, students who initially chose "trial and error" did the same as those producing a data set. And those who produced data actually found points, which in theory should lie on the graph representing the area function. What the strategies have in common is how to find the exact value of the least area – the optimization problem. In all cases the calculations are not simple, though they are manageable. This creates the need to start talking about other methods for optimization problems, especially in cases where we end up with higher degree polynomials.

Further examples of applying these ideas and other principles to design IBMT based modules are given in other MERIA project publications (see http://www.meria-project.eu/).

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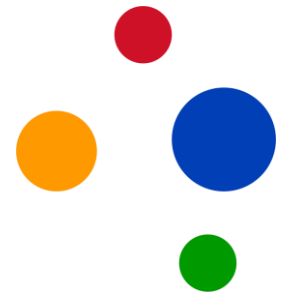
4. Realistic Mathematics Education

Introduction

As mentioned in previous chapters, Artigue and Blomhøj (2013) point out how a number of well-established research programmes in mathematics education have developed methods and ideas for what is now called IBMT. Realistic Mathematics Education (RME) is one of the most prominent, along with TDS.

RME consists of ideas and principles for shaping the learning process. This chapter gives an overview of the main ideas in RME, aimed at teachers and educational designers. Ideas are illustrated by example tasks. In this text the theory of RME is build up from two central principles:

(1) Mathematics is a human activity.
(2) Meaningful mathematics is built from rich contexts.

In the final sections we describe the connection between RME principles and Inquiry Based Mathematics Teaching (IBMT) and discuss RME ideas that can help with the design of IBMT scenarios.

Structuring mathematics

Mathematical knowledge can be structured to very high extent, while RME claims that the learning process requires a less formal approach. In the formal approach, one begins from axioms, postulates and definitions, and from there derives lemmas and theorems. Proofs establish the truth of these propositions within the axiomatic frame. The tradition of organizing and presenting mathematics results in this formal way stretches from Euclid (300 BC) to contemporary mathematics research. Mathematics as a building, where axioms are foundations and logic mortar, is impressive and effective. The formal presentation of results enables unambiguous academic communication. No wonder some have based mathematics education on it. In many countries, geometry was taught from *the Elements* of Euclid until the 1950's. In the 1950's and 1960's, the New Math movement introduced set theory as a basis for secondary mathematics education.

Mathematics as a human activity

*S*hould this highly structured body of mathematical knowledge be the leading inspiration on how we shape mathematics education? RME takes a different point of view. Its leading inspiration is that mathematics *is a human activity.* The organised body of mathematical knowledge is a product of this activity. For example, a good definition of a mathematical object is often the result of a long process of mathematical thoughts, ideas and attempts. RME underlines the importance of these processes that lead to the polished version of a mathematical object or result.
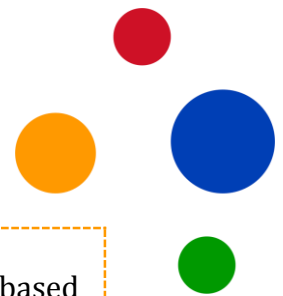
One could, in the very beginning of a chapter on logarithms, define the logarithmic function as the inverse of the exponential function. An RME based approach would rather begin with a task that shows the need for the concept. The exercise should allow the students to experience this necessity for a logarithmic function themselves. Here is a basic idea.

Rogier puts 100 euro's in the bank. The interest rate is 2%. Fill in the table.

| Amount (A)	100	≈ 108,24	≈ 129,36	≈ 199,99	≈ 507,24
Years passed (t)	0				

Do you know a function to compute t from A?

Of course the answer to the last question is likely to be "no", but it is important for learners to ask this question and realise there is need for a new function. Learners are not used to this type of questions. For this reason the question might better be answered in a classroom discussion guided by the teacher. Learners may come up with (square) root functions and need guidance discovering why that is wrong.

Anti-didactical inversion

Presenting a learner with mathematics in its highly structured (axiomatic system based) version is an *inversion*. The learner is confronted with the result of an often long and difficult process of doing mathematics. If the learner is supposed to study mathematics this way, then the learning process is an inversion of the process that led to the mathematics. He will have to work hard (or to wait) to find out which questions gave rise to this mathematics and which problems were solved by it. The teacher could have consciously chosen for this approach, but RME claims it is not a didactical one: it is an *anti-didactical inversion* (Freudenthal, 1991)*.*

Generally, a formal presentation of mathematics is rather inaccessible for novice learners. There are many didactical arguments against confronting a learner with mathematics in its highly structured polished version in the beginning of the learning process:

- The natural process (being led by questions, problems, curiosity ...) of arriving at the mathematics is not shown. Meaning and motivation is taken away from the learner.
- Intuition that leads to the theory is remote from the learning process.
- It is not clear what is solved, modelled or captured by the system (and what is not).
- Heuristics that were needed to organise the mathematics in that way are neglected.
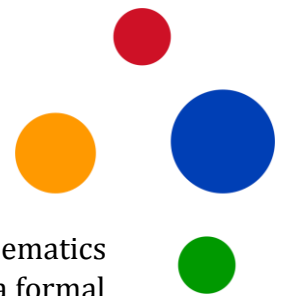
- The presentation can be too dense or sparse. An aspect of the mathematics may be very difficult to grasp but only be given little emphasis in a formal presentation.

> Many mathematicians, including mathematics teachers, will remember being confronted with the ε, δ-definition of limits in the first year of their studies, or even in high school. Why was this so inaccessible? It does not make sense to a learner if she has no understanding of problems with rigorous proof that emerged in Analysis at the beginning of the 19th century. What issue does it solve? Why such effort to prove something obvious? Why do other definitions not work?
>
> Similarly, stating the distributive law "$a \cdot (b + c) = a \cdot b + a \cdot c$" just like that in secondary education, followed by exercises like "*expand* $5 \cdot (a + 2)$", is formally a correct order, but does not convey any meaning to the learner. Neither does it answer the question why this is a useful rule or skill.
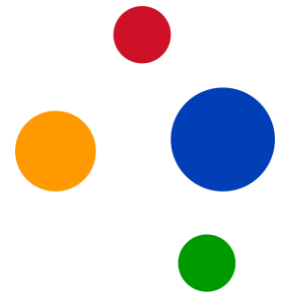
The role of realism in learning processes

Obviously the (formal) meaning of mathematical objects and procedures is carefully and precisely described in formal presentations of mathematics. Since presentations with formal characteristics can be inaccessible and not didactical for a novice learner, how is one supposed to teach this meaning?

A learning process is formed by a set of learning activities. One of the central ideas of RME is that the situations on which those activities are based should be real or realistic. The meaning of mathematical concepts and procedures is constructed from what is already meaningful to the learner, from what is *real* to the learner.

What is meant by "real/realistic" in RME? Something is real for a learner, if it has some evident meaning to her, if she can grasp it. Something is *real* for a group of learners when it is *common sense* to them. 'Real" does not (necessarily) mean "modelled on reality", for example modelled on situations from other disciplines, like physics or economics. Nor does a "realistic" learning situation necessarily mean that it is based on an everyday life experience. And "real" is definitely not meant ontologically: what does and what does not exist. In fact, "meaningful mathematics" might be a better expression than "realistic mathematics", but the latter happens to be the label, as it emerged in the previous century. Meaningful mathematics is learned by starting from what is already meaningful for the learner, in particular from meaningful contexts. As Freudenthal states it:

> *How real [the] concepts are depends on the conceiver, and under given circumstances cognitive grasps can be more vigorous than manual and sensual ones, which are in fact always mixed up with cognition" and "(What is real is) mutually connected by actual, imagined and symbolised relations (….) which can extend from the nucleus of everyday life experience to the far frontiers of mathematical research, depending on the involvement of who is concerned.* (Freudenthal, 1991, p.30).

The distributive laws can be introduced in a realistic geometric context:
Compute the area of the whole rectangle in two ways:

(1) First the dark, then the light rectangle and then add the two
(2) First compute the whole width and then multiply by the height

(based on v*an den Broek et al., n.d.*)
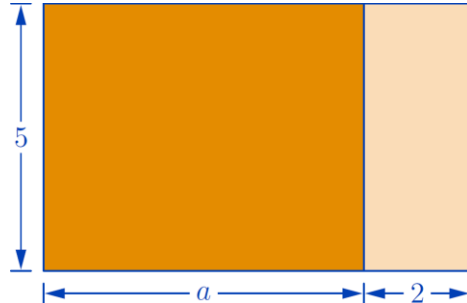
Why is this a (more) realistic approach? The learner is assumed to be familiar with computing areas. The meaning of the equality emerges naturally as the outcomes of the two computations have to be equal. Meaning emerges from the task alone. The teacher has a role introducing the task, guiding the learners and reflecting on the task in classroom. He has to embed the task in a learning process in the right way. Later in this text follows more on RME views on this.

Rich structures and rich contexts

According to RME new meaning of mathematics for a learner is not drawn from the formal mathematical edifice, but mostly from what is real for the learner. The didactical situation should allow development of new knowledge from what is already meaningful. This means it should be rich in non-mathematical contexts and in mathematical structures. Here are possible ways in which a mathematical structure or a context can be *rich:*

(1) it connects to various aspects of the learner's common sense - the more connections made, the richer the structure;
(2) its usefulness carries further mathematically than the situation where it is introduced;
(3) it allows different approaches or solutions on different levels.

We now proceed to illustrate these ways by concrete examples.
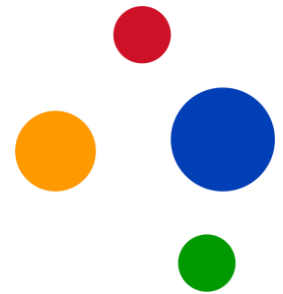
Point (1) is illustrated by the following 'The tower and the bridge' task. It was used in an experiment for introducing scale and geometrical reasoning in a 3D context (Goddijn, 1979).

Below you see two photos of the same beautiful Dutch landscape with a tower and a bridge from different viewpoints. Which is higher: the tower or the bridge?
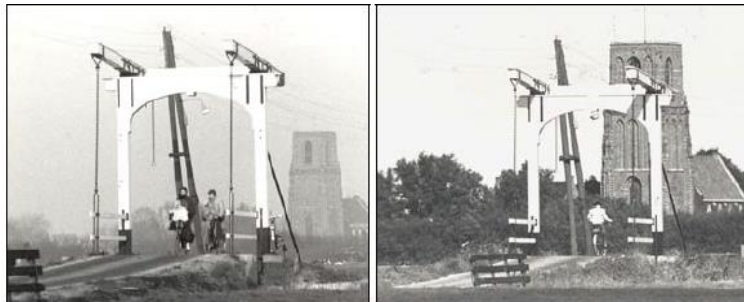

Dutch school children ride their bikes everywhere, in particular to school. Surely, they will have seen bridges and towers like this in relative positions. With their smartphones they take photos (and edit them) daily. Moreover, everyone has an innate ability to imagine scenes from different perspectives. So this situation is realistic in many ways. And now they are required to think about it mathematically. They will have to introduce notions like viewpoints, projections, vision lines and scaling to discuss the situation, which is the goal of the task.

The Figure below summarizes some of the mathematical aspects of the problem. The photos are depicted in a more correct relative scaling.
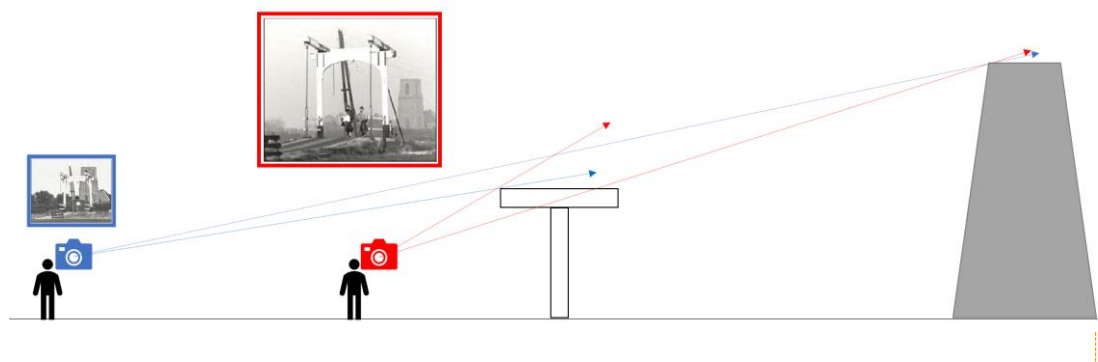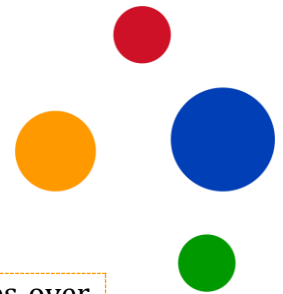
Point (2) is illustrated again by the rectangle example above. It carries over nicely to exercises like: expand $3 \cdot (x + y + 3)$, where the rectangle is divided in three instead of two. It also applies to $(a + 7)(b + 8)$, where the rectangle is divided in four
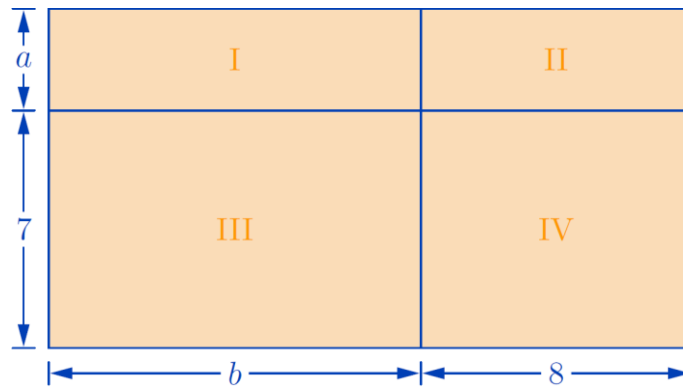

This, in contrast, is sometimes explained with another model that does not satisfy point (2). That second model is called a "parrot beak" and it is illustrated like this:

$$(a + 7)(b + 8) = a\,b + 8a + 7b + 56$$

As soon as you multiplied two terms, you draw the line between them. If you have done it well, the beak appears. This model is a mnemonic technique and provides no understanding of what is happening. It does not satisfy (2), since you only get a beak expanding $(a + c)(b + d)$, not with $(a + c)(b + d + e)$ or more complex expressions.

If one focuses on the formal presentation of mathematics as an inspiration for education, then it is a natural choice to begin with the mathematical objects with least structure. This way you build mathematical knowledge up from its fundamental notions. Geometry would begin from axioms on point and lines. Analysis could begin from sets, natural numbers to real numbers, then functions, etcetera. Such an approach was used during New Math in the 1960's. But this is another incarnation of the anti-didactical inversion. Most of these structures are the end point of a process of abstraction, "impoverishing" and reorganisation of mathematical knowledge. According to RME it is more instructive for the learners to go through a process that achieves this themselves.
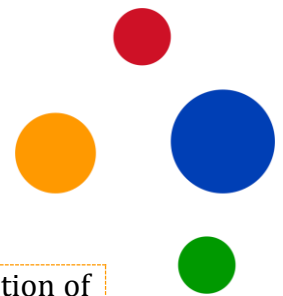
Point (3) can be illustrated by the following exercise. Just after substitution of numbers for variables is introduced one can proceed to solving equations.
Find solutions for:

$$2x = 8$$
$$7 + x = 15$$
$$x^2 = 25$$
$$x + 8 = 2x + 2$$
$$(x + 2)^2 = 16$$

This list could be much longer; the more variety in equations, the richer the task. Without having previously learned any solution methods the success of learners will vary. They will also apply various types of reasoning. Through this exercise the teacher can find out what comes naturally to the learners and use this in a later stage, when formal solution methods are discussed. The teachers find out about the differences between the learners.

Mathematising

RME promotes mathematics as a human activity. Freudenthal calls one of main components of this activity *mathematising*:

> *Mathematising is the entire organising activity of the mathematician, whether it affects mathematical content and expression, or more naive, intuitive, say lived experience, expressed in everyday language… (The goal is) offering non-mathematical rich structures in order to familiarise the learner with discovering structure, structuring, impoverishing structures and mathematising. By this means he may discover the powerful poor structures in the context of the rich ones in the hope that, by this approach, they will also function in other (mathematical as well as non-mathematical) contexts. Starting with poor mathematical structures may mean that one will never reach the rich non-mathematical ones, which are in fact the proper goal.* (Freudenthal, 1991, p.31 and p.41)

Mathematising involves: axiomatising (creating an axiomatic mathematical system), formalising (the transition from an intuitive to a formal approach), schematising (forming meaningful networks of concepts and processes), algorithmising (the transition from solving a problem by hard work to solving it by routine), modelling (building schemes that represent, idealise, simplify other schemes), etcetera.

One can distinguish two directions in mathematising: horizontal and vertical (Treffers, 1987). Horizontal mathematising is the transition of a problem or situation into a mathematical discourse. It enables the *mathematical* treatment or discussion of the situation. Vertical mathematising is mathematising within a mathematical discourse.

As soon as one poses (and answers) questions about a situation in terms of quantities, distances, shape, symmetry, order, probability or other type of structures studied within mathematics, horizontal mathematising takes place. Both types of mathematising should be practiced by learners. If the horizontal part is neglected, then the learner loses the connection between mathematical knowledge and the situations where it is applied. If the vertical process is neglected the learner misses the opportunity to form the deep connections within mathematics, build the formal system and find a better understanding.
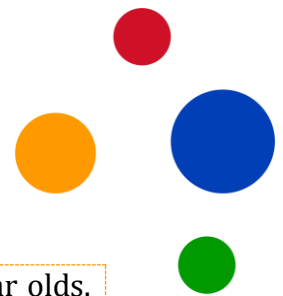
This task is part of a course material on discrete models for 16/17 year olds. The goal of the task is to exercise modelling skills with sequences, to practice sum sequences and to introduce the geometric series. It begins by introducing the famous paradox of Achilles and the turtle. Many students are familiar with it, but they can easily be brought into difficulties trying to disentangle it (e.g. in a classroom discussion).

Achilles and the turtle are in a footrace. Since Achilles is faster, the turtle has a head start. Unfortunately, each time Achilles reaches the place the turtle was a moment before, the turtle has already progressed a bit. This way Achilles can never overtake the turtle and the turtle wins the race. What's wrong with this reasoning? How can we solve the paradox by mathematical reasoning?
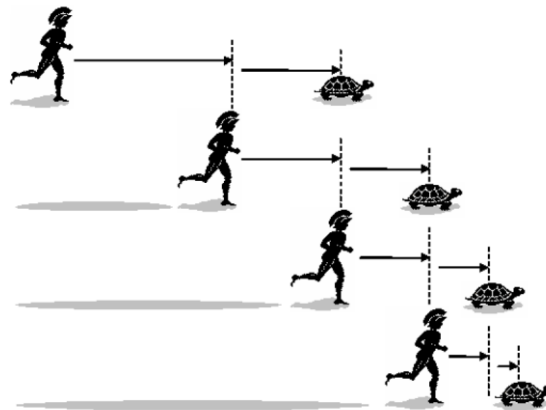
The students are then challenged to model the situation (as a mathematical sequence). This naturally leads to questions on the role of time and distance as variables.

A possible answer begins with some assumptions, say: the head start is 1, Achilles' speed is 1 and the turtle's is ½. Then the distance between at the times Achilles reaches the turtle's previous position the is modelled by a sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$$

The total distance covered by Achilles and the amount of time passed at each of those moments are modelled by a sequence

$$1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \dots$$

But how to deal with infinite sequences? If you add up an infinity of numbers is the outcome not infinity? This is the heart of the paradox! The answer lies in the geometric series, which is a major learning goal of the task.
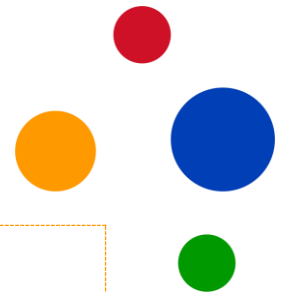
In a follow up exercise students study the picture:

Informal reasoning with this picture gives students a way to compute the geometric series solving the paradox.

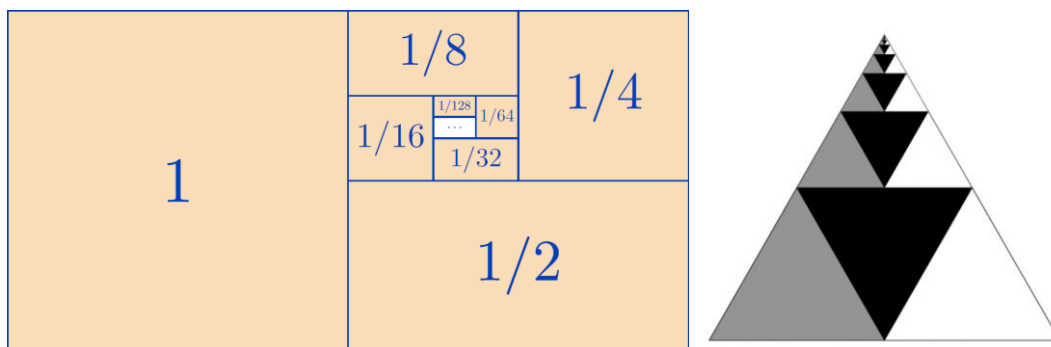Then follows a process of vertical mathematising. the student is challenged to find a similar result for the picture on the right and then to formalise and generalise what is represented visually in these pictures to

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}.$$

Finding the expression $\frac{1}{1-x}$ is a big challenge.

After applying this result to other intriguing situations, like $0,9999 \ldots = 1$ (a nice example of a mathematical context), the students' interest in finding a proof should have been stimulated. In the proof pseudo-formal techniques are used

$$(1 - x)(1 + x + x^2 + x^3 + \cdots) = 1 + x + x^2 + x^3 + \cdots - x - x^2 - x^3 + \cdots = 1.$$

Later this can be further formalised in notation by introducing limits and $\sum$-notation.

This sketch of a learning scenario shows examples of modelling and formalising, starting from the rich context of a famous paradox and accessible pictures. Note the order of the activities; the learner is enabled to arrive at the more formal result through studying the concrete contexts.

Horizontal mathematising from rich contexts to tie the bonds with reality

RME is very concerned with the bonds of mathematics with reality. As Freudenthal (1991, p.81) puts it:

> *The world is noisy; mathematising the world means looking for essentials, sensing the message within the noise. This, too, has to be learned, that is, reinvented by the learner, and the earlier the better; once the learner has fully been indoctrinated by ready-made schemes and algorithms it may be too late.*

Next to "mathematics as a human activity", "bonds with reality" is one of the main focusses of RME. To stimulate those bonds learning activities should involve a
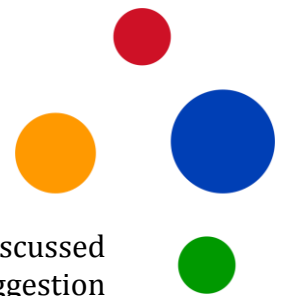
sufficiently (non-mathematical) rich context. Earlier in the chapter we discussed rich contexts. Let us elaborate on this with a few suggestions. Each suggestion should, of course, meet the criteria for richness mentioned before.

    a) A location. For example, a stockroom or a music festival
    b) A story, such as the paradox of Achilles and the turtle described above.
    c) A human activity. For example designing a house, or flying a plane.
    d) News or a historical event. For example, statistical claims in a newspaper.

The following exercise comes from *De Wageningse Methode* (van den Broek et al., n.d.). It is part of a chapter about matrices. A large part of the chapter revolves around the rich context of a car sales company. It has a headquarters and a branch. It sells cars of type A, B and C. The car stock is represented by a matrix S

$$\begin{array}{c} \\ Headquarters \\ Branch \end{array} \begin{array}{ccc} A & B & C \\ \begin{pmatrix} 15 & 13 & 7 \\ 3 & 4 & 11 \end{pmatrix} \end{array}$$

In previous exercises learners have been adding matrices to adjust to stock. Now a value matrix V is introduced (in thousands of Euros)

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{ccc} sale & cost & profit \\ \begin{pmatrix} 12 & 11 & 1 \\ 30 & 28 & 2 \\ 20 & 17 & 3 \end{pmatrix} \end{array}$$

The total sale value of the cars in the headquarters is
$$15 \cdot 12 + 13 \cdot 30 + 7 \cdot 20 = 710 \text{ (thousand Euros).}$$

    a) Compute the total sale value of the cars in the branch.
    b) Compute the total cost value of the cars in the headquarters. And in the branch.
    c) Compute the total profit value of the cars in the headquarters. And in the branch.
    d) Use the totals you found in a), b) and c) to fill in a totals matrix T

$$\begin{array}{c} \\ Headquarters \\ Branch \end{array} \begin{array}{ccc} sale & cost & profit \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{array}$$

Then follows an explanation that what one has done is actually a kind of multiplication for matrices $S \cdot V = T$, and T is defined as the product matrix. The benefit of this approach is that the operations performed for matrix multiplication come naturally and in a meaningful way; thanks to a well-chosen context.

Emergent models

So how does a learner arrive at more formal mathematical knowledge in RME? In work of Streefland (1985), Treffers (1987) and, later, Gravemeijer (1994), a special role is given to *models* as they arise in the mind of learners. In their work, models are mental schemes of concepts and processes related to a situation. From horizontal mathematising a model *of* a situation emerges. This model represents the learner's informal mathematical activity with respect to the situation. It gives
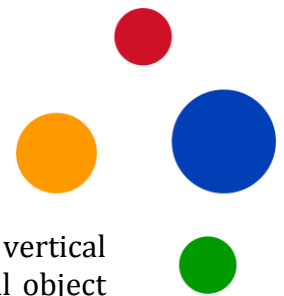
meaning to the situation for a learner. From here a process of vertical mathematising can take place: building a (more abstract) mathematical object from a concept, or an algorithm from a process. The new model is a more formal one. After one or more of such steps it is not a model *of* a specific situation, but a model *for* a class of situations enabling mathematical activity without reference to the situation that gave rise to the model. However, if needed, one could give meaning to the model by linking through the intermediate models all the way back to the original one. This is one of the reasons why RME prefers to work with models that carry further than the situation where they arise (see point (2) about rich exercises).

The gradual emergence of a formal model may stretch over a long period of education. As an example let's look at the emergence of the concept of a *function* (Doorman, Drijvers, Gravemeijer, Boon and Reed 2012). We take as a starting point that the learner is familiar with the concept of a variable, including substitution of a value for a variable. Exercise for 12 year olds (adapted from De Wageningse Methode, cf. van den Broek et al., n.d.):

Look at the scheme on the right.
Make the table with the numbers 1, 2, 3, 4, 5 and 10.

Sam finds an outcome of 10. What was his starting number?

And with 343?
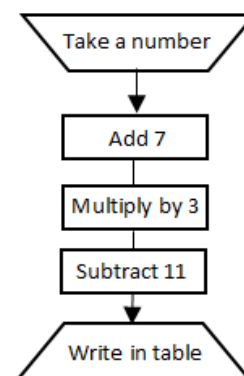
This informal activity will later progress into the use of formulas to represent the arrow scheme. Learners will work with such computational schemes and formulas and they will gradually form a reality for the learner.

At some point new basic computations are added: the sine, cosine and tangent, denoted $\sin(x)$, etc. Learners do not learn how the computation is accomplished (in general), but just what their geometric meaning is. This is an important shift of point of view. The next step is that in analogy a new notation enters: $f(x)$, where f represents a computational scheme. At this point the computational scheme itself becomes an object. Learners will have to study properties of the object, like the domain or derivative. But the concept of function is introduced based on a transformation of models: a model *for* the concept of function, based on models of functions, and not based on a definition. An actual formal definition of a function is reached from a different path altogether: set theory!
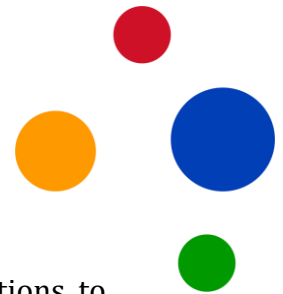
Guided reinvention

Horizontal mathematising activity opens up a situation, or a class of situations, to mathematical discourse. Through vertical mathematising, models of informal mathematical activity are gradually transformed into models representing formal mathematical knowledge. One could say that in this way the formal mathematics is *reinvented* by the learner. This process can in many cases not be the same as the original invention. The way a professional mathematician arrived at a result may use motivation and knowledge not available to a learner. The challenge for the RME-teacher is to facilitate a process that is suitable for the learner. The process has to be *guided*. As written in Freudenthal (1991) "Inventions, as understood here, are steps in learning processes, which is accounted for by the "re" in reinvention, while the instructional environment of the learning process is pointed to by the adjective "guided"". In addition to the previous discussion one can add the following arguments in favour of guided reinvention (Freudental 1991):

1. Knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others.
2. Discovery can be enjoyable and so learning by reinvention may be motivating.
3. It fosters the experience of mathematics as a human activity.
4. It ensures the mathematical approach fits the level of the learner.

The reinvention principle should be put in the perspective of the central claim of RME with which we started this discussion: that mathematics education is not just about the body of mathematical knowledge, but also about learning to mathematise. Therefore the process of reinvention is valued as much as the outcome.

Guiding towards inventions

How to guide learners towards their reinventions? "Guiding means striking a delicate balance between the force of teaching and the freedom of learning" (Freudenthal 1991). Obviously the guided activities should promote horizontal and vertical mathematising. The aim should be that the learners themselves produce solutions to set problems, and perhaps even produce new problems.

The teacher's instruction should promote discussions between learners themselves and between learners and the teacher. Discussions among learners allow them to test, focus and reformulate ideas without a teacher directing them to a desired outcome. Not all learners will mathematise and invent at the same speed. Discussions help learners to align their ideas.

If the teacher is involved in discussion, then learners benefit from his attempts to go along with their reasoning to help them see where it might lead. The reason for this is that learners own approaches are based on what is meaningful for them. If the teacher can guide those methods to an acceptable solution, then the odds of the learner understanding the solution increase.
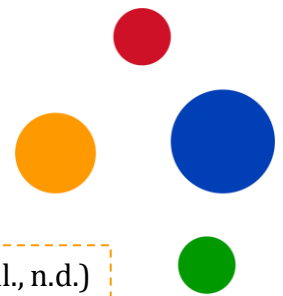
This exercise adapted from De Wageningse Methode (van den Broek et al., n.d.) aims at a reinvention of the *completing the square* method. In the right picture the L-shape is completed to a square.
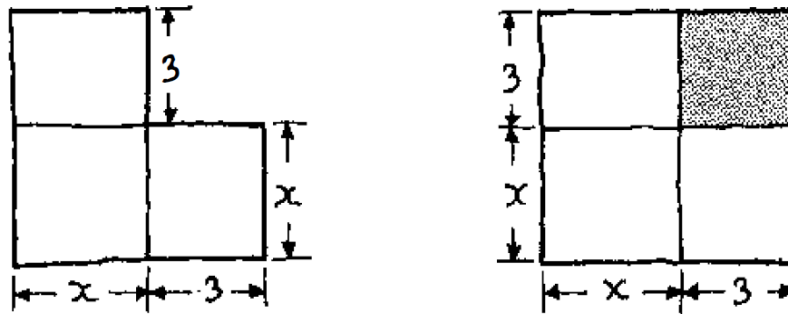
a) Write an expression in x for the area of the L-shape in the left picture.
b) What is the area of the grey square?
c) What is the length of the side of the big square?
d) Explain how (a) and (b) lead to the equality $x^2 + 6x = (x + 3)^2 - 9$.
e) Check this equality by expanding the brackets in the right expression.
f) Sketch an L-shape with area $x^2 + 10x$.
g) What equality can you derive from this L-shape?

This exercise is repeated with different numbers (introducing fractions as well), but leaving the choice of drawing an L-shape to the learner. The important observation here is that the reinvention of the algorithm is left to the learner. The learner is supposed to do the algorithmising.

A learner's own invention (such as a concept, algorithm, model, or a way to solve -a problem) may not be the most efficient or beautiful one. It may be different from the one the teacher had in mind as a desired learning outcome. At the end of a reinvention activity the teacher could try to formulate a joint outcome in a classroom discussion. The teacher should take care to connect the outcome to the learners' contributions.

RME and IBMT

Where is the common ground between RME and IBMT? A central concept of IBMT is *inquiry:* a process similar to how mathematicians and scientists work when confronted by a new phenomenon.

> *Many daily-life phenomena can be described, investigated and understood with the help of mathematics in combination with science or common sense, and are therefore a rich source for IBMT[2]... (*Artigue and Blomhøj, 2013*)*

RME and IBMT have some principles in common. Both theories describe how daily-life situations form a rich source for learning. They advocate knowledge construction through methods inspired by science and knowledge construction:

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[2] In this booklet we use the term IBMT, where Artigue and Blomhøj use IBME.
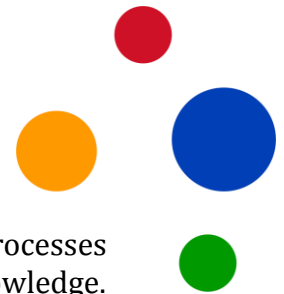
inquiry, discovery or (re)invention. Both IBMT and RME describe these processes as social: learners working together to rediscover and reconstruct knowledge. RME emphasizes that reinvention will be different from invention, since the knowledge that forms the starting point for a specialized researcher and for a novice learner are very different.

In addition to traditional roles, the teacher acquires a new one in RME and IBMT: he is a facilitator and a guide of inquiry and mathematising. The learners and their ideas play the central role. The teacher helps to formalize the informal approaches of the learners, as discussed before.

RME and IBMT view the proficiency in inquiring and mathematising themselves as learning goals, in addition to domain knowledge. This is a significant shift away from exclusively domain knowledge centered approaches.

RME-structure for IBMT-modules

So far we have discussed various aspects of RME, with several example tasks. To conclude we sketch an outline how to string tasks together into a learning trajectory, for example a module.

1. An introduction: present a context with a relatively open problem (possibly for the students to discover or formulate). This problem is going to be overarching for the entire module. It will be approached in various mathematical ways.
2. A phase of horizontal mathematising: mathematical language is introduced to discuss the situation. The learners form a first informal model of the situation.
3. A phase of vertical mathematising: the mathematics is involved in the problem is further developed. The model is made more abstract, more general.
4. Conclusion and reflection: the learner reflects on the whole process, integrates ideas, makes acquired metacognitive skills explicit, the learners share their findings, the teacher guides and highlights main learning points.

In each phase there are elements of inquiry: the discovery and/or formulation of the problem, forming a first informal model, abstracting, sharing findings. The challenges involved in applying these ideas and other principles to design IBMT based modules are addressed in other MERIA project publications (see http://www.meria-project.eu/).
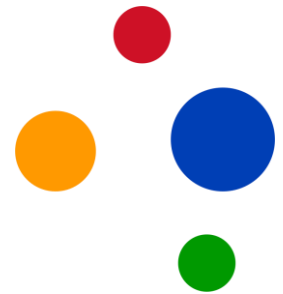
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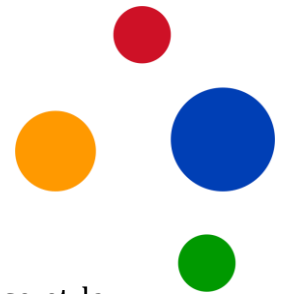
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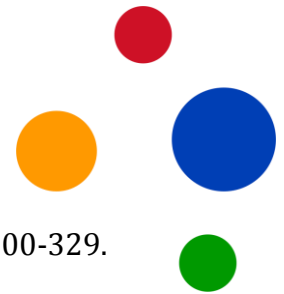
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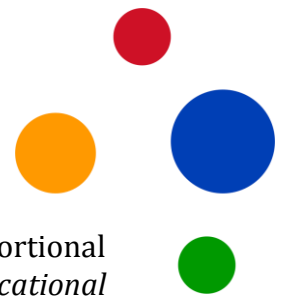
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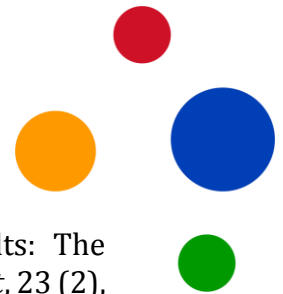
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Appendix. An outline of Key References: suggestions for further reading related to the MERIA project.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM – The International Journal on Mathematics Education, 45, (6), pp. 797-810*.
The paper argues how IBE/IBME invites students to "work in ways similar to how mathematicians and scientists work". They start by presenting Dewey as a philosopher who strived to overcome the distinction between knowing and doing by viewing human behavior as reflective inquiry. They further list by whom Dewey was inspired. They list elements of inquiry practice which seems crucial: reflective inquiry mixes induction and deduction, process concerning daily life and scientific activity, hands-on activities and that IBE should develop the students' habits of mind in the direction of those underlying inquiry processes. The descriptions of inquiry from the PRIMAS and Fibonacci projects are described and how they relate to the idea of progressive development of "big ideas". The migration of IBE to mathematics education is argued as relating to Polya's "How to solve it" and more recent theories and approaches to the teaching of mathematics. Hereafter, a short presentation of these theories and approaches are given and how they relate to IBE. The approaches treated are: The problem solving tradition, the Theory of Didactical Situations, Realistic Mathematics Education, Modelling perspectives (from Mathematical Competence Theory), the Anthropological Theory of Didactics and the Dialogical and critical approaches. While summing up the authors argue that teachers need to have experience and to exercise inquiry in mathematics themselves in order to teach inquiry based and it is suggested to differ between "inquiry by teachers and inquiry in teaching" and the latter seem to require considerable collaboration among teachers for IBME to be realized in classrooms. As concluding remark the authors list ten concerns, which should be taken in to consideration when engaging in IBME and which are addressed with different weight on each concern when teaching is designed based on the existing approaches to mathematics teaching presented earlier in the paper.

Artigue, M. & Baptist, P. (2012). Inquiry in Mathematics Education , *Resources for Implementing Inquiry in Science and in Mathematics at School.* **Retrieved from http://www.fibonacci-project.eu**
This part of the booklet from the Fibonacci project describes previous and present attempts to teach mathematics in an inquiry based manner. The Fibonacci project continues some of the ideas from the German SINUS, which defined features involved in inquiry processes in mathematics teaching. The first part of the booklet section points out what approaches to math education known from the literature capture IBME features. Inquiry in science often draws on already sensed experiences, which can be further studied in cyclic processes, which do not apply to the case of mathematics. Here the cumulative nature of the discipline is a challenge. Hence the design task is different if we want to ensure that students reach a certain learning goal, which again links to already developed knowledge
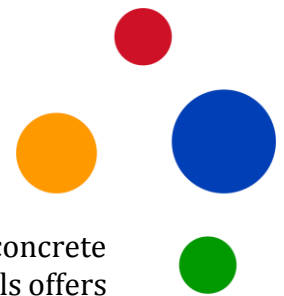
within the students and form the basis for more formal proving of the concrete ideas developed during the inquiry activity. In this context ICT or CAS-tools offers special opportunities and challenges when designing IBME activities – examples are provided of different ICT designs. In the first half it is briefly argued what elements: Modelling, RME, ATD, TDS and critical approaches and problem-solving can offer IBME. But also the obstacles one might encounter when implementing it in school systems is presented in this booklet section.

In the second part a more practical (teacher) perspective is given on the IBME. From a characterization of standard teaching it is pointed out, how teaching should be altered: what should the teacher do less and more of? What actions should the students engage in and how do teachers make them do that? It is argued how these actions support the students' development of problem-solving and metacognitive competences. Finally examples are given on IBME tasks with and without computers.

Artigue, M., Dillon, J., Harlen, W., & Léna, P. (2012). Learning through inquiry, *Re-sources for Implementing Inquiry in Science and in Mathematics at School.* Retrieved from http://www.fibonacci-project.eu/resources
More general on the ideas of the Fibonacci project not restricted to mathematics

Artigue, M., & Houdement, C. (2007). Problem solving in France: didactic and curricular perspectives. *ZDM – The International Journal on Mathematics Education, 39*, 365–382
The paper gives an overview of how problem solving can be regarded and approached from the point of view of TDS, ATD and "conceptual fields". A few examples are given, of how problem solving is articulated in curricula at different levels of mathematics. Most of the results presented relate to the change of focus with respect to problem solving in curricular reforms from 1945 to 2002. Changes in curricular reflect the changed role of primary education. It is described through examples how didactical research has influenced the curricular changes with respect to problem solving through design centers as IREM, which provide the support of in-service teachers to realize the intended changes. However, there are still problems when studying the realized curriculum in the classrooms, where teachers find definitions of a problem blurred and they have difficulties navigating in open processes and tend to put equal value to different answers of varying quality. It is suggested that stronger links between research and practice as well as teacher training will improve the realized curriculum.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990*. Dordrecht: Kluwer Academic Publishers.
The book presents most of the Theory of Didactical Situations, which has been developed by Guy Brousseau, and further developed together with his research group. TDS is introduced through the example of "The race to 20". The analogy between learning and winning a game becomes clear in the introduction and a first presentation of the phases of action, formulation and validation is given. Chapter 1 starts by a presentation of what *didactique* is in French research,
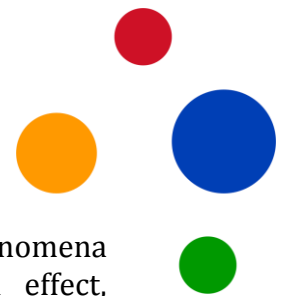
concerning the objects and phenomena which are studied. Among the phenomena are some unintended effects of teaching: Topaze effect, Jourdain effect, metacognitive shifts and improper use of analogies. Further, the notions of didactical situation, adidactical situation and the didactical contract are presented. Examples are given on devolution of an adidactical situation and further paradoxes regarding the didactical contract are discussed. The paradoxes relates to students adjustment to situations and the learning potentials of doing that. In the last part emphasis is put on how the phases and situations can be modelled through the design of milieu, which leads to formulation of intended learning if the students adapt to the milieu of the situation.

Chapter 2 continues the design element by presenting the notion of epistemological obstacles, problem and what didactical engineering is from the point of view of TDS. The chapter relates to problem situations and Brousseau's study regarding the teaching of decimals. Further a distinction between what obstacles can be dealt with in classrooms and what obstacles are external to the classroom is given.

Chapter 3 provides an analysis of the possible outcomes of the teaching of decimals in French primary school from 1960s and 1970s based on previous curricular and approaches to teaching. This is continued in chapter 4, where conclusions on the mathematical, epistemological ad the didactical analysis are drawn. Based on these design examples other examples are presented and discussed: the pantograph and the scaling of drawings, the puzzle task moving from an additive to multiplicative domain, decimal numbers and the rational numbers. Next, the analysis of a situation is presented, which covers the design of a situation where the thickness of a piece of paper is determined and the analogy of the learning situation with a (didactical) game.

Chapter 5 elaborates on the notion of didactical contract both in relation to design issues and in relation to the effects on students learning. It relates to the phases of the didactical game with an emphasis on the knowledge to be taught in the designed situation.

The last chapter 6 addresses the relevance of TDS research to teacher practice including techniques for teachers and how research knowledge can become reality in the classroom practice.

Burkhardt, H., & Bell, A. (2007). Problem solving in the United Kingdom. *ZDM - The International Journal on Mathematics Education, 39*, **395–403.**
The paper gives a historic overview of political decisions made throughout the last 100 years regarding teaching in mathematics. It is problematized that in recent years policy makers seem to act based on their own experiences with respect to what mathematics teaching is and should be rather than relying on research knowledge. Hence, inquiry approaches to the teaching of mathematics is not emphasizes or supported in the British school system.

Chevallard, Y. (2015). Teaching Mathematics in tomorrow's society: a case for an oncoming counter paradigm. In *The Proceedings of the 12th*
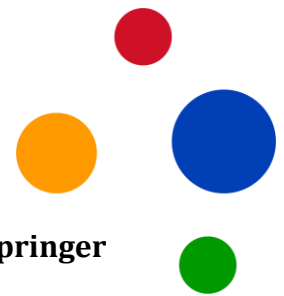
International Congress on Mathematical Education (pp. 173-187). **Springer International Publishing.**

This is a survey paper, introducing elements of the Anthropological Theory of the Didactic (ATD), another French theory of didactics. Ordinary classroom teaching presenting and explaining procedures or formulas is characterized as the paradigm of visiting works. The paper argues that mathematics teaching should head towards a new (counter)paradigm: Questioning the world. It is proposed that teaching should be based on open questions, which students answer by engaging in the study of existing resources and employing newly gained and existing knowledge to answer the open question. In this process students are supposed to derive new questions from the given one. The design tool for this kind of teaching is called Study and Research Paths (SRP) and is pointed out by other researchers (including papers in this list) to be a promising model for IBME.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of Classroom Mathematics Traditions: An Interactional Analysis. American *Educational Research Journal*, **29 (3), 573–604.**

The authors are analyzing two examples of teaching place value numeration in US grade two and three. They introduce a number of notions from American mathematics education literature to analyze the two teaching situations. They identify the situations as school mathematics and inquiry mathematics respectively. They emphasise the different role played by instructions and the verification of students' answers. They mention the work and some notions of Brousseau's TDS, however they do not wish to analyse the two teaching situations using the notion of didactical situations. They conclude that "In addition, we contend that cognitive models which document students' construction of increasingly sophisticated mathematical objects are essential to analyses of their activity as they participate in the interactive constitution of an inquiry mathematics tradition." The paper show an attempt to conceptualise how inquiry like mathematics education can be analysed and compared to traditional approaches. Most of the findings can be related to the notion of didactical contract from TDS, but it is not done in the paper.

Dewey, J. (1902). *The Child and the Curriculum*. **Chicago: University of Chicago Press.**

He discusses how educational systems are arranged in logical structures. However the logic is often the one produced by grownups and is the product of years of dealing with the knowledge to be taught. This might lead to challenges for child and its' learning since it might not fit with the child's experiences. On the contrary teaching should revolve around children's actions and it is concluded: "Action is response; it is adaptation, adjustment. There is no such thing as sheer self-activity possible—because all activity takes place in a medium, in a situation, and with reference to its conditions"

Dewey,J. (1929). *The Sources of a science of education*

Chapter 1: Education as a science. He argues for the need of regarding education as a science, where we share knowledge in a scientific way. Some teachers have a
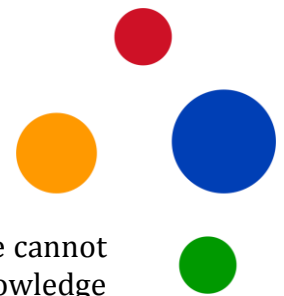
talent for teaching, but if we do not study, what this talent is made of, we cannot share the practice or ideas on teaching. But there is a danger of knowledge gathered as regarding education as science, will be misused as quick fixes by persons in educational systems

Chapter 2: Borrowed techniques insufficient. It is argued that techniques cannot be borrowed from natural sciences. And at this time of history, it is unclear what and how to measure objects in the field of educational research.

Chapter 3: Laws vs. Rules.

Discusses how school systems and knowledge is arranged and why this might fail in teaching and learning for all and the free play of thought, where the latter might actually be central for learning.

Dewey, J. (1938). *Logic: The theory of inquiry*. **New York: Henry Holt and Company, Inc.**

The book discusses inquiry from different perspectives: common sense and scientific inquiry, the structure of inquiry and construction of knowledge, working hypotheses etc. The main emphasis is put on inquiry in science. A chapter is devoted to the mathematical discourse of inquiry, where it is concluded that: "The considerations here adduced have an obvious bearing upon the nature of test and verification (See ante, p. 157). They prove that in the practice of inquiry verification of an idea or theory is not a matter of finding an existence which answers to the demands of the idea or theory, but is a matter of the systematic ordering of a complex set of data by means of the idea or theory as an instrumentality." Hence it is the generality, which can be drawn from the concrete experiment or experience, which is interesting. Different notions and concepts from mathematics (e.g. isomorphic, a relation etc.) are discussed in the context of inquiry and in mathematics and to what extend they do mean the same.

Dorier, J. & Garcia, F.J. (2013). Challenges and opportunities for the implementation of inquiry-based learning in day-to-day teaching. *ZDM - The International Journal on Mathematics Education*, **45(6).**

The paper argues about the conditions and constraints which might favour, or on the contrary hinder, a large-scale implementation of inquiry-based mathematics and science education, on the basis of our work within the PRIMAS project in 12 European countries. The model of the educational system provided by the Chevallard's anthropological theory of didactics (ATD) as a systemic institutional perspective helped in structuring the analysis of conditions and constraints of the systems in these countries. It is a complement to the approach through the analysis of teachers' beliefs and practices (Engeln et al. in ZDM Int J Math Educ 45(6) 2013). In the approach, teachers are actors of institutions, representing some disciplines, embedded in a school system, sharing some common pedagogical issues, are considered in relation to society. The analysis is organized according to four levels of institutional organization that co-determine both content and didactical aspects in the teaching of mathematics and sciences: society, school, pedagogy and disciplinary.

Drobnič Vidic, A. (2011). Impact of Problem-based Statistics Course in Engineering on Students' Problem Solving. *International Journal of Engineering Education* **27(4):885-896.**

Abstract. In this comparative study, we examined the level of basic discipline knowledge and problem-solving abilities in problem-based learning (PBL), incorporated into a traditional curriculum in an introductory statistics course. Progressively less structured, less familiar and more open problems were presented to engineering students. Engineering problems triggered the learning of new statistical contents and activated small group problem solving. Students as a group determined the learning goals, individually searched for information, and together analysed the information collected. Such a problem-solving process with real-world problems is often seen as unstructured and time-consuming. An experiment was carried out to find out whether this approach yields adequate basic statistical knowledge and improves problem solving. Two randomised groups of students from the same engineering programme were compared: one group used PBL and the other followed the traditional method of instruction. The results of statistical analysis showed that engineering students with the PBL approach acquired sufficient basic statistical knowledge and were better able to solve statistical problems from the field of engineering than the students who followed the traditional way of instruction. Some characteristics of the implementation of the course are discussed, as well as some limitations of the study.

Drobnič Vidic, A. (2015). First-year students' beliefs about context problems in mathematics in university science programmes. International Journal of Science and Mathematics Education, 13 (5), pp. 1161–1187.

Abstract: Mathematics-related beliefs play an important role in the willingness to engage in academic activities in mathematics education. Such beliefs might not be consistent with the beliefs students hold about context problems that require sufficient mathematical knowledge and the application of such knowledge to various real-life situations. This study was designed to examine differences between students' mathematics-related beliefs and beliefs about context problems. The variations in these beliefs could explain the different amounts of effort students put into solving context problems on one hand and in solving typical mathematical tasks on the other. The study included 261 first-year students: students in one group were enrolled in academically more demanding study programmes (n = 162), while students in the other group (n = 99) were enrolled in less demanding study programmes. The results revealed significant differences in beliefs between the two groups. A detailed analysis indicates the factors which need to be emphasised when designing problem-based mathematics education to promote the successful problem solving of context problems.

Drobnič Vidic, A. (2016). Using a Problem-Based Leaning Approach to Incorporate Safety Engineering into Fundamental Subjects. *Journal of Professional Issues in Engineering Education and Practice.* **142 (2).**
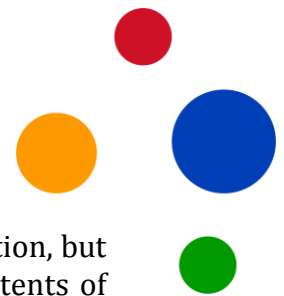
Abstract. Safety is considered as an important area of engineering education, but it is often not addressed adequately in an engineering curriculum. Contents of safety engineering were incorporated in an introductory statistics course through problem-based learning (PBL) approach. Novices were learning statistical contents via PBL problems from the field of safety engineering. They were divided in two groups according to the partial assessment option they chose: the group with classical assessment and the group with assessment of an independent PBL engineering problem that was designed in accordance to the campaign coordinated by the European Agency for Safety and Health at Work. In the problem, students were analyzing the quality of installation of fire extinguishers in more than 200 buildings, as well as their maintenance. The aim of our study was to find out if the assessment of such a problem can be used to assess students' holistic statistical knowledge, if students can get new insights in the field of safety engineering, and if such assessment suits the ABET criteria. Students' questionnaire also gave us information on the students' perception of the difficulty of PBL approach in both assessment options.

Drobnič Vidic, A. (2017). Teachers' Beliefs about STEM Education Based on Realisation of the "Energy as a Value" Project in the Slovenian School System. *International journal of engineering education* (in press).

Abstract. The cross-curricular project Energy as a Value described in this study involved almost all subjects in the K-12 curriculum of the so-called technical gymnasium. It became the framework for an effective Science, Technology, Engineering and Mathematics (STEM) education. Although the project offered interdisciplinary connection of all STEM subjects, promoted problem-based learning and pointed out to applications of subjects' contents to engineering profession it was not added up as a successful one. Teachers' satisfaction was questionable at the end of the four-year project time. Teachers were not initiators for a new project. The Engineering Education Beliefs and Expectations Instrument for STEM education is used in order to find the reasons for such an ambitious project not being carried out again. The instrument documents teachers' beliefs and expectations about pre-college engineering instruction, college preparation, and career success in engineering, and to compare teachers' views. It is applied to teachers of technical gymnasiums in Slovenia that teach STEM subjects in order to find out if there are differences between beliefs of teachers that carried out the Energy as a Value project and teachers from other technical gymnasiums, as well as differences between beliefs of mathematics / science teachers and technology-based / engineering teachers. The results of statistical analyses give answers about obstacles that teachers who carried out the ambitious STEM education in a particular school system might be confronted with.
Keywords: STEM education; teachers' beliefs; K-12 curriculum; interdisciplinary engineering project; project-based learning.

Elia, I., Gagatsis, A., Panaoura, A., Zachariades, T., & Zoulinaki, F. (2009). Geometric and Algebraic Approaches in the Concept of "Limit" and the
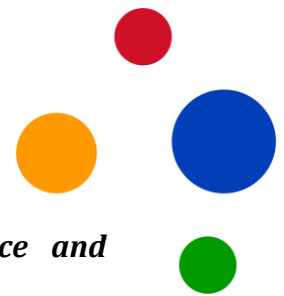
Impact of the "Didactic Contract". *International Journal of Science and Mathematics Education*, 7 (4), 765–790.
This paper reports on a study with a large number of upper secondary students' engagement in problems regarding the concept of limit, where the students were supposed to change freely from the algebraic to the geometric domain and back again. To what extend students succeeded in the none-routine problems requiring a change of domain depended on the degree by which the students were bound by a traditional didactical contract.

Ellerton, N. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: Development of an active learning framework. Educational Studies in Mathematics, 83, 1, pp. 87-101.
The paper starts arguing for the importance of being able to question the content, which you are supposed to learn and learn, to question existing knowledge requires creativity and imagination and is how advances are made in science. Therefor this should be promoted in the teaching. The paper sketches some designs created by pre-service lower secondary teachers have designed teaching activities engaging students in posing problems.

Engeln, K., Euler, M., & Maaß, K. (2013). Inquiry-based learning in mathematics and science: a comparative baseline study of teachers beliefs and practices across 12 European countries. *ZDM – The International Journal on Mathematics Education*. Advance online publication. http://link.springer.com/journal/11858
The paper presents some of the results of a questionnaire answered by the teachers engaged in the PRIMAS project. It shows that teacher in general have a positive attitude towards IBL, but also that they consider a lack of resources as a major obstacle to implementing IBL. Also national restrictions in the educational system are pointed out as challenging. By contrast, classroom management is not regarded as a major problem by the teachers.

Euler, M. (2011). *PRIMAS survey report on inquiry-based learning and teaching in Europe*
The PRIMAS project showed that in most EU countries at least some teachers in mathematics and science have experience with Inquiry Based Learning (IBL), but there are differences in the interpretation of the notion, hence an IBL lesson can appear very different in one country compared to another. It is suggested that initiatives supporting the implementation of IBL is initiated around teachers, who have some experience already and an interest in pedagogical or didactical issues. The project identified three main factors making the implementation of IBL problematic: classroom management, resources and restrictions from the educational system in specific countries.

García, F. J. (2013) *PRIMAS guide for professional development providers.*
The report lists a number of concrete initiatives for how to teach in-service teachers to use IBL, the theoretical approaches captures modeling, Lesson Study and to fit IBL with local requirements for in-service teacher training. The modules
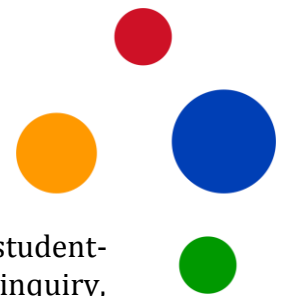
of the PRIMAS in-service teacher training covered the following topics: student-led inquiry, tackling unstructured problems, learning concepts through inquiry, asking questions that promote IBL, students working collaboratively, building on what students already know, self and peer assessment.

Godino, J.D., Batanero, C., Canadas, G., Contreras, J.M. Linking inquiry and transmission in teaching and learning mathematics. In K. Krainer & N. Vondrova (Eds.). *Proceeding of the Ninth Congress of the European Society for Research in Mathematics Education***, 2015, Prague, Czech Republic. pp.2642-2648.**
The paper describes different theories that assume that learning mathematics should be based on constructivist methods where students inquire problem-situations and assign a facilitator role to the teacher (RME, TDS), and contrast them to the theories that advocate for a more central role to the teacher, involving explicit transmission of knowledge and students' active reception. The authors hold the view that mathematics learning optimization requires adopting an intermediate position between these two extremes models.

Gravemeijer, K. & Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32***, 6, pp. 777-796.**
The authors give an account of the main contributions to mathematics education by Hans Freudenthal, who regarded mathematics as a human activity. He continued the idea of guided reinvention (also known from Dewey's work), which questioned the formation of curricula at the time. He wanted to promote the idea of putting processes rather than fixed pieces of content as a central element of what students should learn. As a result mathematics teaching should be based on modeling problems where students mathematize matter from reality, but with no clear intra- and extra mathematical reality. Later a difference between vertical and horizontal mathematization was introduced. Freudenthal criticized the role played by generic theories on pedagogy or learning theories in mathematical education research. Rather he proposed the approach of Realistic Mathematics Education (RME), which is a phenomenological approach to mathematics teaching.

Gueudet, G., & Trouche, L. (2011). Mathematics teacher education advanced methods: an example in dynamic geometry. *ZDM – The International Journal on Mathematics Education, 43* **(3), 399–411.**
An example of how in-service teacher training can support teachers in the design or development of inquiry based teaching employing a dynamic geometry computer program (ICT based IBME). The teachers in this study are teaching at upper secondary level and the theoretical approach is the very recent theory of documentational genesis.

Van den Heuvel-Panhuizen, M. (2000). Mathematics education in the Netherlands: A guided tour. *Freudenthal Institute Cd-rom for ICME9. Utrecht: Utrecht University.*
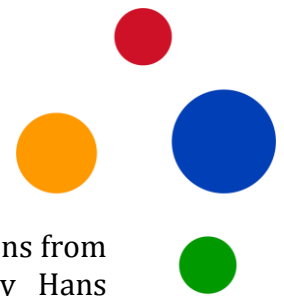
This is a survey paper, which introduces the central constructs and notions from RME and Dutch didactics tradition, starting with contributions by Hans Freudenthal, towards more recent developments. Three primary school examples are provided in the text.

Hersant, M., & Perrin-Glorian, M.-J. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactic situations. *Educational Studies in Mathematics, 59(13)***, 113–151.**
The paper presents some of the challenges when teaching is designed to offer students a larger degree of initiative in the classroom and how that increases the uncertainty of the teacher. By employing the notions from TDS the authors analyse two case studies of teaching, where the authors have had no influence on teaching design or the conduct of the teaching. Based on this the authors discuss the challenges and possibilities for bringing constructivist approaches to teaching into the classroom.

Kilpatrick, J. (2014). History of Research in Mathematics Education. *Encyclopedia of Mathematics Education***, pp. 267-272. Springer publishing.**
The text gives a short overview of how the research field of mathematics education started to evolve, and that this happened much later than the establishment of a practice. Short account of who took the initiative to form institutions (such as ERME, ICMI, IREM and others) where mathematicians and educational researcher could meet and discuss. The ideas of Felix Klein and the relation between research mathematicians' practice, and the teaching and learning of mathematics, are touched upon. Other more recent problems in the field are outlined, such as the actual and potential roles of technology in mathematics teaching. The text presents an overview of research in mathematics education, and therefore does not present specific research in any detail.

Kilpatrick, J. (2008). The Development of Mathematics Education as an Academic Field. In The first Century of the International Commission on Mathematical Instruction (1908-2008). *Reflecting and shaping the world of Mathematics Education***, pp. 25-39**
First an historic overview is provided with the initiation of commissions for the development of mathematics education, where Felix Klein was an important figure. He introduced a reform program based on an alliance between teachers, scientists and engineers. The idea was to change teacher education to change the teaching in the direction of promoting practical instructions and the development of spatial intuition. It is discussed what mathematics is (which is not easily defined by mathematicians) and what education is. Different approaches are presented such as e.g. Nordic pedagogy tradition and the francophone tradition of didactic. It is argued that mathematics as a field of study as well as a practice revolves around teaching. It is through teaching it is promoted and constituted. This leads to the question (considered by others as well) what is and should be the relation between mathematics as a research field and as a discipline to be taught in different school settings.
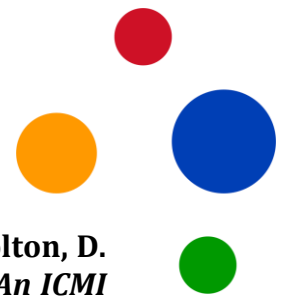
Legrand, M. (2001). Scientific debate in mathematics courses. In Holton, D. (ED.) *The teaching and learning of mathematics at university level: An ICMI study* **(pp. 127-135). Springer Netherlands.**
It is argued that engaging in a mathematics course is not equivalent with students becoming mathematicians, however it might require that they attempt to act like mathematicians and the class form a scientific community debating mathematics. Hence the paper proposes to orchestrate the teaching as a scientific debate. The debate can be initiated as "unplanned" debate based on a question raised by a student, a planned situation with the intention to introduce a new concept or overcome an epistemological obstacle, or the deepening of a concept or theory. Examples are provided of such initiators from first year of university mathematics teaching (including cross disciplinary examples), but several examples might be relevant for the secondary level as well.
For the scientific debate to function it is important that the teacher give enough time for the students to develop their arguments individually, that he/she writes all arguments on the blackboard without judging them and the teacher should strive to maximize the number of students who engage and involve themselves in discovering a rational solution to the problem or conjecture dealt with. The students responsibility is to believe in the conjecture he or she argues for, develop rational arguments for the conjecture and finally to formulate the arguments so convincingly that both fellow students and the teacher is persuaded. In this way the didactical contract of the teaching of mathematics is explicitly changed to one, where the responsibility of students as the one acting, formulating and validating mathematical answers has become explicit. The paper draws on notions from TDS.

Legrand, M. (n.d.) *Les deux ateliers proposés par Marc Legrand reposent sur: Le "Débat scientifique" en cours de mathématiques.* **Retrieved from: http://kordonnier.fr/IMG/pdf/legrand.pdf**
The text provides further arguments regarding the haw scientific debate changes the didactical contract in the teaching and how mathematical activity (of mathematicians) resonates with scientific debate. Further comments from students are provided. Some of those find it difficult to imagine Scientific debate being introduced in primary education, although they found the teaching enlightening and good. Many students find the debates time consuming in the sense, that they are concerned if a Scientific debate course will actually cover the curriculum.

Margolinas, C. & Drijvers, P. (2015). Didactical engineering in France; an insider's and an outsider's view on its foundations, its practice and its impact, *ZDM - The International Journal on Mathematics Education, 47***(6).**
The paper discusses the notion of didactical engineering which has influenced and characterized contemporary research in mathematics education in France. In the paper, the following from an insider's and an outsider's perspective is addressed: (1) the way this notion is theoretically grounded, (2) the kinds of design research practices has it led to and is leading to, and (3) the way it relates to the design
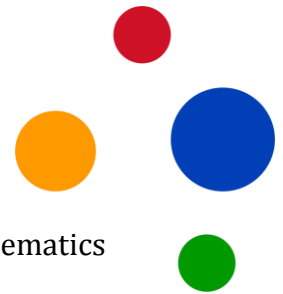
research paradigm. The paper compares the Dutch view on realistic mathematics education and the characteristics of the didactical engineering in France.

Maaß, K. & Artigue, M. (2013). Implementation of iquiry-based learning in day-to-day teaching: a synthesis. *ZDM Mathematics Education, 45,* pp. 779-795
Abstract: This synthesis is designed to provide insight into the most important issues involved in a large-scale implementation of inquiry-based learning (IBL). We will first turn to IBL itself by reflecting on (1) the definition of IBL and (2) examining the current state of the art of its implementation. Afterwards, we will move on to the implementation of IBL and look at its dissemination through resources, professional development, and the involvement of the context. Based on these theoretical reflections, we will develop a conceptual framework for the analysis of dissemination activities before briefly analyzing four exemplary projects. The aim of our analysis is to reflect on the various implementation strategies and raise awareness of the different ways of using and combining them. This synthesis will end with considerations about the framework and conclusions regarding needed future actions.

Miyakawa, T., & Winsløw, C. (2009). Didactical designs for students' proportional reasoning: an "open approach" lesson and a "fundamental situation". *Educational Studies in Mathematics*, 72 (2), 199–218.
The paper analyses and compares two didactical designs on proportional reasoning. The one design is the enlargement of a puzzle known from the literature on TDS. The other design is based on the Japanese tradition of Lesson Study and Open-ended Approach. Both approaches carry an element of inquiry and both share the idea of students learning from potential mistakes.

Monaghan, J., Pool, P., Roper, T., & Threlfall, J. (2009). Open-Start Mathematics Problems: An Approach to Assessing Problem Solving. *Teaching Mathematics and its Applications, 28* (1), 21–31
The paper gives an introduction to problem solving and what defines an Open-start problem, which is characterized by having multiple starting points but only one answer. The paper suggests how these latter problems can be used for assessment purposes, and by changing assessment it is proposed that classroom activities as well will be more inquiry based.

Niss, M. (1999). Aspects of the Nature and state of research in mathematics education. *Educational Studies in Mathematics*, 40, pp. 1-24.
The paper discusses the some fundamental questions for research in mathematics education: what challenges are the educational system facing, and why the teaching of mathematics should be of any interest of research mathematicians. It is formulated in the paper what is meant by a theory, what is mathematics education as a design research and what comes of this kind of research. Several findings are discussed such as perspectives on learning, known obstacles, the role of ICT and the conclusions points towards the need of students develop more
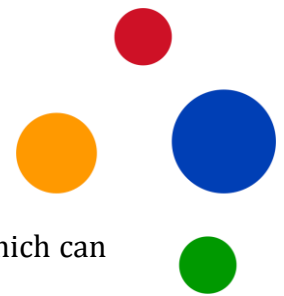
heuristic competences through none-routine mathematical problems, which can be interpreted as more inquiry-based approaches.

Nohda, N. (1995). Teaching and Evaluating Using "Open-Ended Problems" in Classroom. *Zentralblatt für Didaktik der Mathematik*, **27 (2), 57–61.**

Nohda, N. (2000). Teaching by Open-Approach Method in Japanese Mathematics Classroom. *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME),* **(1), 39–53**
An introduction to open-ended approach is given in the paper: based on an initial problem, students' hypotheses and first answers lead to formulate new questions for further inquiry. Examples of different problems are provided, and it is discussed how the teacher deals with the variety of students' answers. The teaching situations are sketched with an emphasis on the communication between the students and the teacher. At the end it is suggested that the link between open-ended approach and modeling should be studied further, and how this kind of teaching affect students' attitudes towards mathematics.

Polya, G. (1945). *How to solve it?* **Princeton, NJ: Princeton University Press.**
This book has been deemed seminal by other researchers in problem solving and IBME as the starting point of the inquiry based approach to teaching and learning of mathematics. Polya describes the processes involved in problem solving as the core activity of a mathematician. He emphasizes the creativity and attitude towards mathematics needed to engage in problem solving activities. He introduces the notion of heuristics in the process of solving problems.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* **San Diego: Academic Press.**
An elaboration and extension of the ideas of Polya. A detailed introduction to what problem solving is, what resources the students are supposed to draw on and what attitudes towards mathematical problems are needed.

Schoenfeld, A. H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning. A Project of the National Council of Teachers of Mathematics* **(pp. 334–370). New York: MacMillan Publishing Company.**
The book chapter gives an introduction to problem solving mentioning Piaget and constructivism, the impact of teachers' epistemological, ontological and pedagogical view on mathematics. He discusses Polya's ideas on heuristics and its relation to metacognition. The paper contains general ideas on how to guide or assist student (university level) in developing problem solving skills and competences. However it is still (in 1992 at least) a challenge how to teach problem solving, since some kind of consensus seem to be reached regarding the definition of what it is.
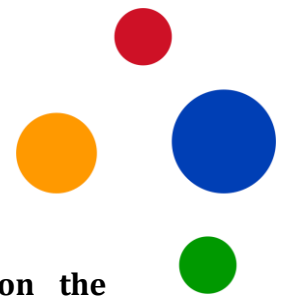
Schoenfeld, A. H., & Kilpatrick, J. (2013). A US perspective on the implementation of inquiry-based learning in mathematics. *ZDM – The International Journal on Mathematics Education*, Volume 45, Issue 6, pp 901–909.
 An discussion of the challenges which implementation of IBMT could face in the United states, considering factors such as current curricula, the capacity of mathematics teachers, and public demands and beliefs concerning the nature and purpose of school mathematics.

Singer, F. M., Ellerton, N., Cai, J. (2013). Problem-posing research in mathematics education: New questions and directions. *Educational Studies in Mathematics*, 83, 1, pp. 1-7.
This is an overview paper introducing the current state of problem posing research in mathematics education. The paper starts by arguing how problem posing support students' development of heuristic competences and how this relates to pursuing ones' own questions. The paper is an introduction to a special issue of ESM and it provides and overview of the approaches to nurture students to pose questions with mathematical content, which can be found in the special issue.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking & Learning*, 10, 313–340.
The paper provides a literature review on classroom discussions, which leads to the presentation of the authors' model involving: anticipating, monitoring, selecting, sequencing, and connecting. It is concluded that: "Thus, the five practices do not provide an instant fix for mathematics instruction. Instead, they provide something much more important: a reliable process that teachers can depend on to gradually improve their classroom discussions over time".

Ulm, V. (2012). Inquiry-based mathematics education in primary school: Overview and examples from Bavaria/Germany. In P. Baptist & D. Raab (Eds.), *Resources for Implementing Inquiry in Science and in Mathematics at School. Implementing Inquiry in Mathematics Education* **(pp.65–81). Retrieved from http://www.fibonacci-project.eu/resources**
An example or model of how to design inquiry based learning environments, followed by some German examples from lower secondary school on basic number theory.
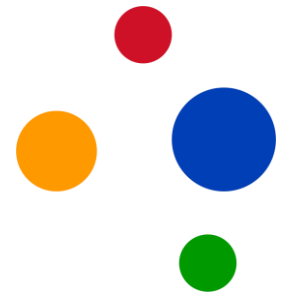
Glossary of special terms used in this booklet

Some of the articles here are based on formulation taken from the internet, which in general is a good source for gaining an initial impression of what a term means. They are presented here for the convenience of the reader and should not replace the necessary in-depth study of the material outlined in the bibliography.

Philosophy of Learning and Knowledge

Epistemology – in the narrow sense, a branch of philosophy concerned with the theory of knowledge, the nature of knowledge, its justification, and the rationality of belief. In mathematics education, epistemological aspects are more broadly concerned with the structure specific parts of mathematics and the obstacles and difficulties it present to learners, as a consequence of this structure.

Constructivism – a philosophical viewpoint about the ways in which humans learn. The formalization of constructivism is generally attributed to Jean Piaget (1896-1980), a famous Swiss psychologist who spent parts of his career to conduct clinical studies of how children learn basic mathematics, among other areas. He modelled human knowledge as built up by mental schemes of various types, and postulated that the construction of those schemes (learning) may take place through what he called assimilation and accommodation of existing schemes to the learner's experience. Constructivist teaching is based on the belief that learning occurs as learners are actively involved in a process of meaning and knowledge construction, as opposed to passively receiving information. Learners are the makers of meaning and knowledge.

General education (including both jargon and vague terms)

Approach in education – a set of principles for teaching and, in a broader sense, a way of interacting with students that facilitates their learning. It can be described in terms of an established theory in mathematics education, or more informally by listing principles based on beliefs about the nature of mathematical knowledge and its learning.

Teaching method – comprises the principles and methods used for instruction, to be implemented by teachers to achieve the desired learning by students. These methods are determined partly by the subject matter to be learned (e.g. quadratic equations), and partly by what is assumed or known about the learner (e.g. familiarity with square roots, interest in the topic, ability to concentrate and work independently, etc.). Teaching methods include lecturing, guidance, and orchestrating student work (with classroom discussion, group projects, working in pairs etc.)

Learning outcome – an expectation about student's knowledge or skills obtained after learning. Such expectations are often quite implicit. We consider that teachers *use learning outcomes* only if they are explicit about it – for instance in preparation, classroom delivery and assessment.

Traditional education – a term (not an approach!) that refers to long-established customs that were used in schools for a long time, often without explicitly stating them. Some forms of education reform promote the adoption of alternative education practices, for instance promoting an increased focuses on individual students' needs and self-control. Many reformers claim that they oppose traditional teacher-centered methods which focused on rote learning and memorization. In fact, "traditional" is often used with little precision.

Passive learning – a method of learning or instruction where students receive information from the instructor and internalize it, often through some form of memorization or rote learning, often with no feedback from the instructor to the learner.

Rote learning - memorization techniques based on repetition. The idea is that one will be able to quickly recall methods or facts, the more one repeats them. Rote learning is usually presented as insufficient and opposed to alternatives with attractive names such as meaningful learning, associative learning, and active learning.

Active learning – learning which is based on learners' own actions and initiative, including a participation in the organisation and evaluation of their learning.

Student-centered learning (learner-centered education) – what is supposed to be achieved by methods of teaching that shift the focus of instruction from the teacher to the student. The idea is to develop learner autonomy and independence by leaving more responsibility for the learning path to students.

Inquiry-based learning – a form of active learning that emerges from answering or posing questions, problems or scenarios – rather than simply acquiring established facts or pursuing a well-trodden path to knowledge. The process is often assisted by a facilitator. Inquirers will identify and research issues and questions to develop their knowledge or solutions. Inquiry-based learning includes problem-based learning, and is generally used in small scale investigations and projects, as well as in research.

Discovery learning – a technique of inquiry-based learning, sometimes presented as a constructivist approach to education. Discovery learning takes place in problem solving situations where the learner draws on his own experience and prior knowledge and is a method of instruction through which students interact with their environment by exploring and manipulating objects, wrestling with questions and controversies, or performing experiments.

Scaffolding (instructional scaffolding) – support given during the learning process which is tailored to the needs of the student with the intention of helping the student achieve his/her learning goals. It combines provision of support (resources, compelling tasks, and guidelines), giving advice and providing
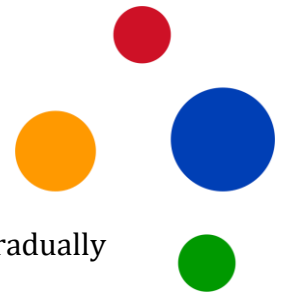
coaching. As in the construction of buildings, the support (scaffold) is gradually removed as students develop autonomous learning strategies.

Heuristic – any approach to problem solving, learning, or discovery that employs a practical method not guaranteed to be optimal or perfect, but sufficient for the immediate goals.

Insight – understanding of the cause and effect within a specific context or a sudden discovery of the correct solution following incorrect attempts based on trial and error. Solutions associated with insight are supposedly more solid than non-insight solutions.

Aha! moment (Eureka effect) – refers to the common human experience of suddenly understanding a previously incomprehensible problem or concept. In some cases, intuition and memory are involved in such effects, which are however generally somewhat inexplicable.

Understanding – relation between the knower and an object of understanding. In general, understanding is a convenient but quite vague term; for a teacher, "understanding calculus" may be used as a quick way to characterize a performance satisfying more explicit criteria, and in general, being more precise about "understanding" is an important objective of theoretical frameworks on education and learning.

Problem solving – reaching a goal in a situation when the proper path to use or a solution is not automatically recognized by the learner. Some question may be a problem to one learner (who does not know any immediate solution method) but not to another (who does know such a method). In other words, problem solving may occur under certain conditions related to the learner.

Problem-based learning (PBL) – a student-centered pedagogy in which students learn about a subject through the experience of problem solving.

Theory of Didactical Situations

Institutional knowledge (sometimes called *public, shared* or *official knowledge*) – knowledge presented in textbooks, journals and resources, which represents a synthesis or the result of different mathematical activities. Easy to observe as it is explicit. Some languages have a specific term for institutional knowledge – for instance, in French, it is called *savoir.*

Personal knowledge (sometimes called *individual knowledge*) – knowledge that students construct while interacting with a mathematical problem (milieu). Often difficult to deduce from observations as it may be tacit, especially in the case of individual work. Some languages have a specific term for personal knowledge – for instance, in French, it is called *connaissances.*
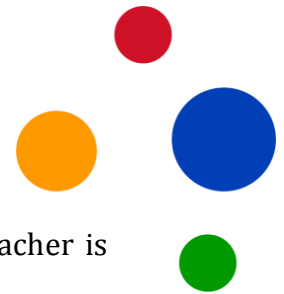
Didactical situation – a teaching and learning situation in which the teacher is explicitly the moderator.

Didactical milieu – the environment with which the student interacts to obtain new knowledge. It consists of the problem, artefacts such as pen and paper, ruler, calculator, CAS-tool (Computer Algebra Systems), a puzzle etc.; in didactical situations it involves also input from the teacher and other students. Learning is modelled as the adaptation, by students, of their personal knowledge to the didactical milieu.

Adidactical situation – interaction of the students with the milieu (a mathematical problem) without the teacher's interference.

Target knowledge – a mathematical statement, method or notion that the teacher sets as the learning aim for her students, in a didactical situation. (A didactical situation is always a situation *for* something – namely, a target knowledge, known by the teacher but, initially, not by the students.)

Devolution phase – phase in which the teacher hands over the milieu to students. Devolution refers to the transfer of responsibility to students for solving the problem, or at least attempting to do so. Sometimes several devolutions are deemed necessary to achieve a target knowledge; however, this should be done in controlled ways, to avoid trivializing or fragmenting the problem unnecessarily, as this may reduce in achieving less than the target knowledge (see also *Didactical contract*).

Action phase – phase in which students autonomously engage with a problem.

Formulation phase – phase in which the students explicitly formulate outcomes of the action phase (initial ideas, hypotheses or strategies to solve the problem, more or less general solutions).

Validation phase – phase in which the students are testing their strategies or hypotheses against the milieu, in order to establish the validity of their methods and solutions.

Institutionalisation phase – phase in which the teacher directly declares the institutional knowledge. In some forms of teaching, such as lecturing, it may occur all by itself. In other forms, like the designs often developed within TDS, it is closely related to the preceding phases, so that personal knowledge achieved by the students is merely reformulated in this phase, and explicitly recognized as consistent with official knowledge, warranted by the (school) institution.

Didactical contract – the set of mutual expectations between teachers and students, concerning their respective responsibilities in a concrete didactical situation (or part thereof). The contract is usually implicit and we can only observe its effects in teachers' and students' actions. Some of these effects are
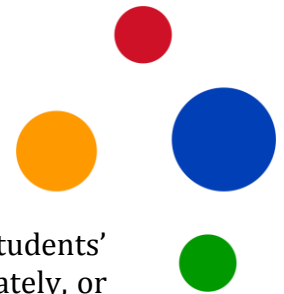
quite general and frequent in mathematics teaching, for instance the students' insistence that teachers must provide answers they do not find immediately, or teachers' tendency to comply with this insistence in more or less concealed ways, for instance by providing "hints" or by reducing the original task. TDS names and studies some of the most common effects, and it is of great interest to both teachers and researchers to be familiar with this classification; interested readers are referred to Brousseau (1997), Chapters 1 and 5.

Didactical engineering - a research methodology based on the controlled design and experimentation of teaching sequences and adopting an internal mode of validation based on the comparison between the *a priori* and *a posteriori* analyses of these. However, since its emergence in the early 1980s, the expression didactical engineering has also been used for denoting development activities, referring to the design of educational resources based on research results or constructions and to the work of didactical engineers. (Source: Encyclopedia of Mathematics Education).

Realistic Mathematics Education

Realistic situation – refers to a situation that is "real" to the learner, in the sense that it concern objects, notions etc. which are familiar to the learner. The situation makes sense to the students and makes them feel comfortable to start thinking because it relates to their prior knowledge. It may be related to every-day (real) life, but that is not necessary.

Rich (structure or context) – allowing for different approaches or solutions, connecting to various aspects of the learner's knowledge, useful beyond the situation where it is introduced.

Mathematisation - entire organizing activity of a mathematician that involves creating axiomatic systems, formalizing, forming meaningful networks of concepts and processes, constructing algorithms, representing and simplifying etc.

Anti-didactical inversion – taking the end point of the mathematician's work as a starting point for teaching mathematics.

Emergent modelling – creation of mental schemes of concepts and processes in the mind of a learner related to a problem situation. Models of informal mathematical activity develop into models for mathematical reasoning.

Guided reinvention – the process in which students reconstruct and develop a mathematical concept in a problem situation with the support (guidance) from books, peers or a teacher.

Horizontal mathematisation – transition or modelling of a real-world problem into a mathematical discourse.
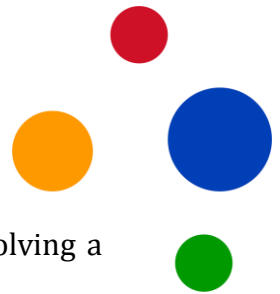
Vertical mathematisation – development of a method or a theory for solving a mathematical problem.

Didactical phenomenology – the art of finding phenomena, contexts or problem situations that beg to be organized by mathematical means and invite students to develop targeted mathematical concepts.